Expansion in “Doubled” Graph

- G k-regular, $|U| = |V| = n$, $|E(\tilde{G})| = k^2 n$.
- for $u \in U$, let $d(u)$ be the number of neighbors of u in S

Double-counting the number of edges between S and U:

$$k|S| = \sum_{u \in U} d(u).$$

Hence, using Cauchy–Schwarz inequality:

$$|E(\tilde{G}[S])| = \sum_{u \in U} d^2(u) = \Big(\sum_{u \in U} d^2(u)\Big) \cdot \Big(\sum_{u \in U} 1/n\Big)$$

$$\geq \Big(\sum_{u \in U} d(u)/\sqrt{n}\Big)^2 = k^2|S|^2/n = \delta^2 k^2 n = \delta^2 |E(\tilde{G})|.$$

Small Set Expansion and Large Eigenvalues

- Since A is symmetric, eigenvectors $v_1, \ldots, v_n$ form an orthonormal basis.
- $1_S = \sum_{i=1}^n c_i v_i$, where $c_i = \langle 1_S | v_i \rangle$.

$$|S| = \|1_S\|_2^2 = \sum_{i=1}^n c_i^2$$

$$1 - \delta \leq 1 - \Phi(S) = 1_S^T A(1_S/|S|) = \frac{1_S^T}{|S|} \sum_{i=1}^n c_i \lambda_i v_i = \frac{1}{|S|} \sum_{i=1}^n c_i^2 \lambda_i$$

$$\leq \frac{1}{|S|} \left[\left(\sum_{i=1}^n c_i^2 \right) - \gamma \sum_{i:\lambda_i < 1-\gamma} c_i^2 \right] = 1 - \frac{\gamma}{|S|} \sum_{i:\lambda_i < 1-\gamma} c_i^2$$

- Let $w = \sum_{i:\lambda_i \geq 1-\gamma} c_i v_i$.

$$\|1_S - w\|_2^2 = \left\| \sum_{i:\lambda_i < 1-\gamma} c_i v_i \right\|_2^2 = \sum_{i:\lambda_i < 1-\gamma} c_i^2 \leq \frac{\delta}{\gamma} |S| = \frac{\delta}{\gamma} \|1_S\|_2^2$$

Expansion in Noisy Hypercube

See lecture notes 9.