2SAT

- Standard algorithm.
- MAX-2SAT is hard:
  - $(0.954\ldots + \varepsilon)$-approximation NP-hard
  - $(0.943\ldots + \varepsilon)$-approximation hard assuming UGC
  - There exists a polynomial-time $0.943\ldots$-approximation algorithm.

Maximum Acyclic Subgraph

- Order vertices arbitrarily (ordering $\prec$).
- $G_1 = (V, \{(u, v) \in E : u \prec v\})$ (“left to right”)
- $G_2 = (V, \{(u, v) \in E : v \prec u\})$ (“right to left”)
- Both G_1 and G_2 are acyclic and

$$\max(|E(G_1)|, |E(G_2)|) \geq \frac{1}{2}(|E(G_1)| + |E(G_2)) = \frac{1}{2}|E|.$$

Independent Set

Reduction from 3SAT φ to IS instance G_φ with n vertices:

- For each clause $c = x \vee y \vee \neg z$, create vertices (c, x), (c, y), $(c, \neg z)$ forming a triangle.
- Add all edges of form (c_1, x), $(c_2, \neg x)$.

$$\alpha(G_\varphi) \leq n/3$$

Independent set A gives an assignment:

- If for some c, we have $(c, x) \in A$, set $x = \text{true}$, otherwise set $x = \text{false}$.

If A is maximal, then

$$|A| = \text{number of satisfied clauses}$$

- $\text{OPT}(\varphi) = 1 \Rightarrow \alpha(G_\varphi) = n/3$
- $\text{OPT}(\varphi) \leq \theta \Rightarrow \alpha(G_\varphi) \leq \theta n/3$

$\text{GapIS}_{1/3, \theta/3}$ is NP-hard (apx. factor $1/\theta$).

Independent Set – the power graph

For $G = (V, E)$, let G^k be the graph with $V(G^k) = V^k$ and

$(u_1, \ldots, u_k)(v_1, \ldots, v_k) \in E(G^k)$ iff for some $i \in \{1, \ldots, k\}$, $u_i v_i \in E(G$

Observe:

- $A_1, \ldots, A_k$ independent sets in $G \Rightarrow A_1 \times \cdots \times A_k$ independent set in G^k.
- $X = \{a, b, \ldots\}$ independent set in $G^k \Rightarrow \{a_i, b_i, \ldots\}$ independent set in G for $i = 1, \ldots, k$.

Corollaries:

- Maximal independent sets in G^k are products of independent sets in G.
- G^k has at most $\left(2^n\right)^k = 2^{nk}$ maximal independent sets.
- $\alpha(G^k) = \left(\alpha(G)\right)^k$.

For fixed k, $|V(G^k)| = n^k$ is polynomial,

- $\alpha(G) \geq \beta n \Rightarrow \alpha(G^k) \geq \beta^k n^k$, $\alpha(G) \leq \gamma n \Rightarrow \alpha(G^k) \leq \gamma^k n^k$

GapIS$_{\beta,\gamma}$ is NP-hard$\Rightarrow$ GapIS$_{\beta^k,\gamma^k}$ is NP-hard (factor $(\beta/\gamma)^k$).

Independent Set – random induced subgraph

For induced subgraph H of G^k, observe

$$\alpha(H) = \max\{|V(H) \cap A| : A \text{ maximal independent set in } G^k\}.$$

Suppose each vertex of G^k belongs to H independently with probability p.

- for each $A \subseteq V(G^k)$, $\mathbf{E}[|V(H) \cap A|] = p|A|$

By Chernoff inequality, for each $A \subseteq V(G^k)$,

- $\mathbf{Pr}[|V(H) \cap A| \geq (1+\varepsilon)p|A|] \leq \exp\left(-\frac{\varepsilon^2}{3}p|A|\right)$
- $\mathbf{Pr}[|V(H) \cap A| \leq (1-\varepsilon)p|A|] \leq \exp\left(-\frac{\varepsilon^2}{2}p|A|\right)$

Independent Set – random induced subgraph

Suppose $\alpha(G^k) = \delta n^k$ for some $\delta > 0$, and $pn^k = n^c$:

- $|V(H)| = (1 \pm \varepsilon)n^c$ a.a.s.
- For largest i.s. B in G^k, $|V(H) \cap B| \geq (1-\varepsilon)\delta n^c$ a.a.s.
- For any maximal i.s. A in G^k,
  $\mathbf{Pr}[|V(H) \cap A| \geq (1+\varepsilon)\delta n^c] \leq \exp\left(-\frac{\delta\varepsilon^2}{3} n^c\right)$

Hence, the probability that $\mathbf{Pr}[|V(H) \cap A| \geq (1+\varepsilon)\delta n^c]$ for any of at most 2^{kn} maximal independent sets is at most

$$2^{kn} \cdot \exp\left(-\frac{\delta\varepsilon^2}{3} n^c\right) \leq \exp\left(kn - \frac{\delta\varepsilon^2}{3} n^c\right) \to 0$$

when $k \ll \delta n^{c-1}$, and thus
$\alpha(H) = (1 \pm \varepsilon)\delta n^c = (1 \pm 2\varepsilon)\delta|V(H)|$ a.a.s.
To (w.h.p.) distinguish between $\alpha(G) \leq \gamma n$ and $\alpha(G) \geq \beta n$ (hard), distinguish between $\alpha(H) \leq (1+2\varepsilon)\gamma^k n$ and $\alpha(H) \geq (1-2\varepsilon)\beta^k n$.

- We need $\delta = \gamma^k$ and $k \ll \delta n^{c-1}$: OK for $k = \log n$ and c sufficiently large.
- Approximation factor $(\beta/\gamma)^k = n^{\log \beta/\gamma}$.