Summer School on Polynomial Paradigm in Algorithms	**Summer 2020**

Background Lecture 4: Matroids

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In lectures 4 and 5 we will talk about matroids and prove several fundamental theorems. Let us first formally define matroids. A *matroid* $M = ([n], \mathcal{I})$ is defined on a ground set of elements, say $[n] = \{1, \dots, n\}$ and a family of *independent sets* $\mathcal{I} \subseteq 2^{[n]}$ that satisfies the following properties:

Downward Closed: If $A \in \mathcal{I}$ then for any $B \subseteq A$, we have $B \in \mathcal{I}$.

Exchange Property: If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there is an element $i \in B \setminus A$ such that $A \cup \{i\} \in \mathcal{I}$.

It follows from the exchange property that all *maximal* independent sets of a matroid M have the same size. Any maximal independent set of a matroid is called a *base* of the matroid.

Given any $S \subseteq [n]$, the rank of S, $r(S)$ is defined as follows:

$$r(S) = \max_{I \subseteq S, I \in \mathcal{I}} |I|,$$

i.e., it is the size of the largest independent set in S. So, if S is an independent set, then $r(S) = |S|$.

Matroids were defined and studied by Whitney around one hundred years ago in order to generalize the notion of linear independence in vector spaces. It is not hard to see that for any set of vectors $v_1, \dots, v_n$ over a field F we can define a matroid, where any sets $A \subseteq [n]$ is an independent set if the corresponding set of vectors are linearly independent. The notion of rank in this case is the same as the rank of the vector space defined by of $v_1, \dots, v_n$. Such a matroid is called a *linear* matroid.

Another famous example of matroids is the *graphic matroid*. Here, $[n]$ is the set of edges of a graph G and a set of edges form an independent set if they *do not* induce a cycle. It is not hard to see that graphic matroids are special cases of linear matroids. If G is connected, then bases of its graphic matroid are exactly spanning trees of G.

Bases Generating Polynomial. Given a matroid $M = ([n], \mathcal{I})$ of rank r, the *bases generating polynomial* of M is defined as follows:

$$g_M(z_1, \dots, z_n) = \sum_{B:\text{ base of } M} z^B,$$

where as usual, $z^B = \prod_{i \in B} z_i$. One of the goals of this course is to study properties of this polynomial.

Kruskal's Algorithm. Given a matroid M and a weight function $w : [n] \to \mathbb{R}_{\geq 0}$ we can run the following Greedy algorithm (which is analogue of the Kruskal's algorithm) to find the maximum weight base of M: Sort elements of M with respect to w and without loss of generality assume $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $S = \emptyset$. For $i = 1 \to n$, if $S \cup \{i\} \in \mathcal{I}$ then set $S \leftarrow S \cup \{i\}$.

As we see in the next paragraph in fact we can optimize any convex function over the convex combination of bases of M.

Matroid Base Polytope. Given a matroid M, the *matroid base polytope*, $\mathcal{P}_M$ is the convex hull of the indicator vectors of all bases of M. In other words, it is the Newton polytope of g_M. Edmonds proved a simple nice characterization of the matroid base polytope:

$$\begin{aligned} &\sum_{i\in S} x_i \le r(S), && \forall S \subseteq [n] \\ &\sum_{i=1}^{n} x_i = r(M), && \\ &x_i \ge 0, && \forall 1 \le i \le n. \end{aligned} \tag{4.1}$$

Note that the above linear program has exponentially many constraints. But it has an efficient separation oracle, i.e., given any $x \in \mathbb{R}^n$ we can check in polynomial whether x is feasible and if not exhibit a violating constraint in polynomial time[1]. Because of that we can minimize any convex function over this polytope.

Although most of the matroids that we know can be represented by a linear matroid over a field F, it is proved that almost all matroids are not linear [Nel18]. Nonetheless, one can define and study many geometric structures based on this abstract structures. That is why Rota calls matroids *combinatorial geometries*.

Gelfand, Goresky, MacPherson and Serganova [Gel+87] proved the following characterization of the matroid base polytope:

Theorem 4.1 ([Gel+87])**.** *For any integer $1 \le k \le n$, given a k-homogeneous set system $\mathcal{B} \subseteq 2^{[n]}$, i.e., $|S| = k$ for any $S \in \mathcal{B}$, $\mathcal{B}$ is the set of bases of a matroid iff every edge of the polytope conv$\{\mathbf{1}_B : B \in \mathcal{B}\}$ is parallel to $\mathbf{1}_i - \mathbf{1}_j$ for some $1 \le i < j \le n$.*

In other words, the above theorem shows that every edge of the matroid base poltyope $\mathcal{P}_M$ is of the form $\mathbf{1}_i - \mathbf{1}_j$ corresponding to exchanging elements i, j between two bases of M, and vice versa, any homogeneous 0/1 polytope whose edges have this property corresponds to bases of a matroid.

Note that if we had defined matroids as the discrete objects which are representable over a field (i.e., only linear matroid case) then the above theorem would not be true. So, the above theorem shows that in a geometric sense matroids are the right generalization of linear vector spaces. We don't prove this theorem in this note and leave it as an exercise.

Bases Exchange Graph. Given a matroid $M = ([n], \mathcal{I})$ consider the following simple walk on 1-skeleton of the matroid base polytope $\mathcal{P}_M$: Construct a graph $G_M = (\mathcal{B}, E)$ with a vertex corresponding to each base of M and two bases B, B' are connected by an edge if there is an edge between them in $\mathcal{P}_M$. By Theorem 4.1, B, B' are connected by an edge in G iff $|B \Delta B'| = |B \setminus B'| + |B' \setminus B| = 2$. Mihail and Vazirani conjectured that this graph has expansion 1 for any matroid M:

Conjecture 4.2. *Let $G_M = (\mathcal{B}, E)$ be the bases exchange graph of a matroid M. For any $S \subseteq \mathcal{B}$,*

$$h(S) = \frac{|E(S, \overline{S})|}{|S|} \ge 1.$$

In this course, we will try to sketch a proof of this conjecture. One can simulate a random walk on G_M which converges to the uniform stationary distribution over $\mathcal{B}$. An immediate consequence of the above conjecture is that such a walk would mix to the stationary distribution in polynomial time, thus one can generate a uniformly random base of any matroid M.

[1] This follows from the fact that we can optimize any linear function over the bases of M.

Closure Properties of Matroids. Given a matroid $M = ([n], \mathcal{I})$ of rank r, it is closed under many operations.

- Contraction: For an element $1 \leq i \leq n$, M/i is the matroid on elements $[n] \setminus \{i\}$ with independent sets:
$$\{I : i \notin I, I \cup \{i\} \in \mathcal{I}\}.$$
For example, if M is a graphic matroid, then this operation exactly corresponds to edge contraction in graphs.
- Deletion: For an element $1 \leq i \leq n$, $M \setminus i$ is the matroid on elements $[n] \setminus \{i\}$ with independent sets:
$$\{I : i \notin I, I \in \mathcal{I}\}.$$
For example, if M is a graphic matroid, then this operation exactly corresponds to edge deletion.
- Truncation: For an integer $1 \leq k \leq r$, the truncation of M to k, M_k is the matroid with elements $[n]$ and independent sets:
$$\{I : |I| \leq k, I \in \mathcal{I}\}.$$
For example, if M is a graphic matroid, then M_k has all forests of M with at most k edges. Note that the truncation of a graphic matroid is no longer a graphic matroid.

Negative Correlation. Given a matroid $M = ([n], \mathcal{I})$, let μ be the uniform distribution over the bases of M. In this course we study properties of this distribution. It is well-known that if M is a graphic matroid, then the uniform distribution over spanning trees is negatively correlated, namely for any pair of elements i, j (correspond to two distinct edges),
$$\mathbb{P}[i]\,\mathbb{P}[j] \geq \mathbb{P}[i, j]\,.$$

Unfortunately, the negative correlation property does not extend to all matroids. A well-known counter example is the matroid $S8$. Here you can see a representation of this matroid of GF(2) where each element is a column of the following matrix:
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Note that the rank of this matrix is 4; so $\text{rank}(S8) = 4$. It is not hard to see that $|B_1| = 28$, $|B_8| = 20$, $|B_{1,8}| = 12$ and $|B| = 48$, where by B_1 we mean the set of bases that have column 1 and B is the set of all bases. The matroid is not negative correlated because
$$28 \cdot 20 \not\geq 12 \cdot 48.$$

References

[Gel+87] I. Gelfand, R. Goresky, R. MacPherson, and V. Serganova. “Combinatorial geometries, convex polyhedra, and schubert cells”. In: *Advances in Mathematics* 63.3 (1987), pp. 301–316 (cit. on p. 4-2).

[Nel18] P. Nelson. “Almost all matroids are nonrepresentable”. In: *Bulletin of the London Mathematical Society* 50.2 (2018), pp. 245–248 (cit. on p. 4-2).