Summer School on The Polynomial Paradigm in Algorithms **Summer 2020**

Lecture 1: Real Stable Polynomials and Strongly Rayleigh Distributions

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

A multivariate polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ is $\mathcal{H}$*-stable* (or stable for short) if $p(z_1, \dots, z_n) \neq 0$ whenever $(z_1, \dots, z_n) \in \mathcal{H}^n$ where

$$\mathcal{H} = \{c \in \mathbb{C} : \Im(c) > 0\}$$

is the upper-half of the n-dimensional complex plane. We say p is *real* stable if all coefficients of p are real. Unless otherwise specified, all polynomials that we work with in this course have real coefficients.

Fact 1.1. *A univariate polynomial $p \in \mathbb{R}[t]$ is real rooted iff it is real stable.*

This simply follows from the fact that the roots of p come in conjugate pairs. So, if p has a root t with $\Im(t) < 0$, we have $\bar{t}$ is also a root with $\Im(\bar{t}) > 0$.

The above definition can be hard to understand; so, instead we discuss an equivalent definition.

Lemma 1.2. *A multivariate polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is real stable iff for every point $a \in \mathbb{R}^n_{>0}$ and $b \in \mathbb{R}^n$, the univariate polynomial $p(at + b)$ is not identically zero and is real rooted.*

For example, $z_1 - z_2$ is not real stable because for $a = (1, 1)$ and $b = (0, 0)$, $z_1 - z_2$ is identically zero.

Proof. $\Rightarrow$: Fix $a \in \mathbb{R}^n_{>0}$ and $b \in \mathbb{R}^n$. If $p(at + b)$ is identically zero, then for $z_j = a_j i + b_j$, $p(z_1, \dots, z_n) = 0$ so p is not real stable. Otherwise, say $p(at + b)$ has a root t with $\Im(t) \neq 0$. Then, since $p(at + b)$ has real coefficients by Lemma 1.2 Background lecture 1, we can assume $\Im(t) > 0$. Write $t = ci + d$; then for

$$z_j = a_j t + b_j = a_j ci + b_j + da_j$$

$p(z_1, \dots, z_n) = 0$ so p is not real stable.

$\Leftarrow$: Suppose p is not real stable; then there exists $(z_1, \dots, z_n) \in \mathcal{H}^n$ that is a root of p. Set $a_j = \Im(z_j)$ and $b_j = \Re(z_j)$ then $a_j > 0$ for all j so $p(at + b)$ is not identically zero and it must be real rooted. But $t = i$ is a root of $p(at + b)$ which is a contradiction. □

See Figure 1.1 for applications of the above lemma.

Let us discuss several examples/non-examples of real stable polynomials.

Linear Functions: A linear polynomial $p = a_1 z_1 + \dots + a_n z_n$ is real stable iff $a_1, \dots, a_n \geq 0$. To see this, note that if all z_i's have positive imaginary value then any positive combination also has a positive imaginary value and thus is non-zero.

Elementary Symmetric Polynomial: For any n and any k the elementary symmetric polynomial $e_k(z_1, \dots, z_n) = \sum_{S \in \binom{n}{k}} \prod_{i \in S} z_i$ is real stable. I leave this as an exercise.

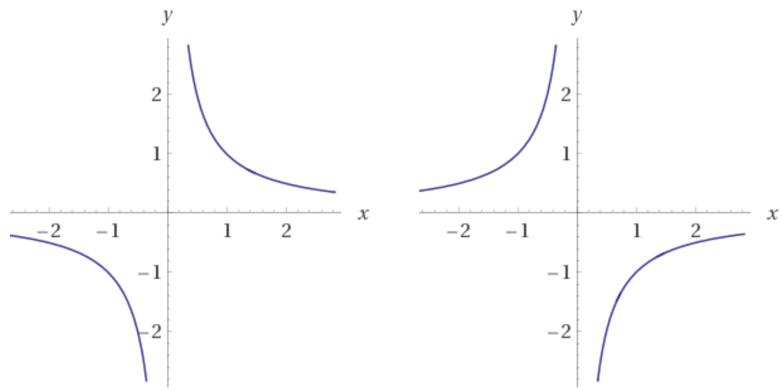

Figure 1.1: Left diagram shows zeros of the polynomial $1 - xy$ and the right diagram shows zeros of $1 + xy$ in the plane $\mathbb{R}^2$. Note that in the left figure any line pointing to the positive orthant crosses the zeros at two points so $1 - xy$ is real stable but this does not hold in the right figure so $1 + xy$ is no real stable.

Non-example The polynomial $z_1^2 + z_2^2$ is not real stable; for example let $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$.

One of the most important family of real-stable polynomials is the determinant polynomial.

Lemma 1.3. *Given PSD matrices $A_1, \dots, A_n \in \mathbb{R}^{d\times d}$ and a symmetric matrix $B \in \mathbb{R}^{d\times d}$, the polynomial*

$$p(z) = \det\left(B + \sum_{i=1}^{n} z_i A_i\right)$$

is real stable.

Proof. By Lemma 1.2, it is enough to show that for any $a \in \mathbb{R}^n_{>0}$ and $b \in \mathbb{R}^n$

$$p(b + ta) = \det\left(B + \sum_{i=1}^{n} b_i A_i + t\sum_{i=1}^{n} a_i A_i\right)$$

is real-rooted. First, assume that $A_1, \dots, A_n$ are positive definite. Then, $M = \sum_{i=1}^{n} a_i A_i$ is also positive definite. So, the above polynomial is real-rooted if and only if

$$\det(M)\det\left(M^{-1/2}\left(B + \sum_{i=1}^{n} b_i A_i\right)M^{-1/2} + tI\right)$$

is real-rooted. The roots of the above polynomial are the eigenvalues of the matrix $M' = M^{-1/2}(B + b_1 A_1 + \dots + b_n A_n)M^{-1/2}$. Since $B, A_1, \dots, A_n$ are symmetric, M' is symmetric. So, its eigenvalues are real and the above polynomial is real-rooted.

If $A_1, \dots, A_n \succeq 0$, i.e., if the matrices have zero eigenvalues, then we appeal to the following theorem. This completes the proof of the lemma. In particular, we construct a sequence of polynomial with matrices $A_i + I/2^k$. These polynomials uniformly converge to p and each of them is real-stable by the above argument; so p is real-stable. $\square$

Lemma 1.4 (Hurwitz)**.** *Let $\{p_k\}_{k\geq 0}$ be a sequence of Ω-stable polynomials over $z_1, \dots, z_n$ for a connected and open set $\Omega \subseteq \mathbb{C}^n$ that uniformly converge to p over compact subsets of Ω. Then, p is Ω-stable.*

Definition 1.5 (d-homogeneous)**.** *A polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is d-homogeneous if $p(\lambda z_1, \dots, \lambda z_n) = \lambda^d p(z_1, \dots, z_n)$ for any $\lambda \in \mathbb{R}$.*

How general are these real stable polynomials and where should we look for them?

Theorem 1.6 (Choe, Oxley, Sokal, Wagner [Cho+02])**.** *The support of any multi-affine homogeneous real stable polynomial corresponds to the set of bases of a matroid (more generally, the support corresponds to a jump system).*

For example, the support of a an elementary symmetric polynomial correspond to the set of bases of a uniform matroid whereas the non-stable polynomial $z_1 z_2 + z_3 z_4$ does not correspond to bases of a matroid.

1.1 Closure Properties

In general, it is not easy to directly prove that a given polynomial is real stable or a given univariate polynomial is real rooted. Instead, one may use an indirect proof: To show that $q(z)$ is (real) stable we can start from a polynomial $p(z)$ where we can prove stability using Lemma 1.3, then we apply a sequence of operators that preserve stability to $p(z)$ and we obtain $q(z)$ as the result.

In a brilliant sequence of papers Borcea and Brändén characterized the set of linear operators that preserve real stability [BB09a; BB09b; BB10]. We explain two instantiation of their general theorem and we use them to show that many operators that preserve real-rootedness for univariate polynomials preserve real-stability for of multivariate polynomials.

We start by showing that some natural operations preserve stability and then we highlight two theorems of Borcea and Brändén.

The following operations preserve stability.

Product If p, q are real stable so is $p \cdot q$.

Symmetrization If $p(z_1, z_2, \dots, z_n)$ is real stable then so is $p(z_1, z_1, z_3, \dots, z_n)$.

Specialization If $p(z_1, \dots, z_n)$ is real stable then so is $p(a, z_2, \dots, z_n)$ for any $a \in \mathbb{R}$. First, note that for any integer k, $p_k = p(a + \imath 2^{-k}, z_2, \dots, z_n)$ is a stable polynomial (note that p_k may have complex coefficients). Therefore by Hurwitz theorem 1.4, the limit of $\{p_k\}_{k\geq 0}$ is a stable polynomial, so $p(a, z_2, \dots, z_n)$ is stable.

External Field If $p(z_1, \dots, z_n)$ is real stable then so is $q(z_1, \dots, z_n) = p(\lambda_1 \cdot z_1, \dots, \lambda_n \cdot z_n)$ for any positive vector $w \in \mathbb{R}^n_{\geq 0}$. If $q(z_1, \dots, z_n)$ has a root $(z_1, \dots, z_n) \in \mathcal{H}^n$ then $(\lambda_1 z_1, \dots, \lambda_n z_n) \in \mathcal{H}^n$ is a root of p so p is not real stable.

Inversion If $p(z_1, \dots, z_n)$ is real stable and degree of z_1 is d_1 then $p(-1/z_1, z_2, \dots, z_n) z_1^{d_1}$ is real stable. This is because the map $z_1 \mapsto -1/z_1$ is a bijection between $\mathcal{H}$ and itself.

Differentiation If $p(z_1, \dots, z_n)$ is real stable then so is $q = \partial p / \partial z_1$. This follows from Gauss-Lucas theorem. If $q(z_1, \dots, z_n)$ is not real stable it has a root $(z_1^*, \dots, z_n^*)$. Define $f(z_1) = p(z_1, z_2^*, \dots, z_n^*)$. Then, $f'(z_1)$ has a root in $\mathcal{H}$. But the roots of $f'(z_1)$ are in the convex hull of the roots of $f(z_1)$ we get a contradiction because the complement of $\mathcal{H}$ is convex.

In the rest of this course we write ∂_{z_1} as a short hand for $\partial p / \partial z_1$.

1.2 Strongly Rayleigh Distributions

Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a probability distribution. For $X \sim \mu$, the *generating polynomial* of μ is defined as follows:

$$g_\mu(z_1, \ldots, z_n) = \mathbb{E}\left[z^X\right] = \sum_{S \subseteq [n]} \mathbb{P}\left[X = S\right] z^S.$$

For example, say B_1, B_2 are two independent Bernoullis with success probabilities p_1, p_2 respectively. Then, their generating polynomial is defined as follows:

$$p_1 p_2 z_1 z_2 + p_1(1-p_2)z_1 + p_2(1-p_1)z_2 + (1-p_1)(1-p_2) = (p_1 z_1 + (1-p_1))(p_2 z_2 + (1-p_2))$$

As a sanity check, observe that $g_\mu(\mathbf{1}) = 1$.

We say μ is strongly Rayleigh (SR) if g_μ is a real stable polynomial. SR distributions were extensively studied in the work of Borcea, Brändén and Liggett [BBL09]. It turns out that closure properties of real stable polynomials translate to closure properties of strongly Rayleigh distributions. Say μ is strongly Rayleigh. Then it remains so under the following operations:

Conditioning In $\mu | i$. This is nothing but $z_i \partial_{z_i} g_\mu$ (up to a normalizing constant).

Conditioning Out $\mu | \bar{i}$. This exactly $g_\mu|_{z_i=0}$.

Projection. Given a set $T \subseteq [n]$, $\mu|_T : 2^T \to \mathbb{R}_{\geq 0}$ is the distribution where for any $A \subseteq T$,

$$\mu|_T(A) = \sum_{S : S \cap T = A} \mu(S).$$

Observe that projection is exactly $g_\mu|_{z_i=1, \forall i \notin T}$.

External Field. Given a non-negative vector $(\lambda_1, \ldots, \lambda_n)$, we define $\mu * \lambda$ as the distribution where

$$\mu * \lambda(S) = \mu(S)\lambda^S.$$

Closure under external fields just follows from the closure of real stable polynomials under external fields, $g_\mu(\lambda_1 z_1, \ldots, \lambda_n z_n)$.

Rank Sequence. The rank sequence of μ is the sequence $a_0, \ldots, a_d$ where $a_i = \mathbb{P}_{S \sim \mu}\left[|S| = i\right]$. It follows that the rank sequence of any strongly Rayleigh distribution corresponds to a sum of independent Bernoullis. This is because $g_\mu(1, \ldots, 1)$ is univariate real rooted polynomial.

1.3 Examples

One of the fundamental examples of strongly Rayleigh distributions is uniform spanning tree distribution. Given a graph $G = (V, E)$, the uniform spanning tree distribution $\mu : 2^E \to \mathbb{R}_{\geq 0}$ has the following generating polynomial:

$$g_\mu(\{z_e\}_{e \in E}) := \sum_T \prod_{e \in T} z_e$$

where the sum is over all spanning trees T of G. In background lecture 2 we prove that

$$g_\mu(\{z_e\}_{e \in E}) = \frac{1}{n^2} \det\left(\mathbf{1}\mathbf{1}^\intercal + \sum_{e \in E} z_e b_e b_e^\intercal\right)$$

Therefore, by Lemma 1.3, μ is strongly Rayleigh. Another example is the uniform distribution over all subsets of size exactly k of $[n]$. The generating polynomial of such a distribution is the k-th elementary symmetric polynomail $e_k(z_1, \dots, z_n)$ thus real stable.

The following lemma is a simple consequence of the above fact:

Lemma 1.7. *Given a graph $G = (V, E)$, let μ be the uniform distribution over all spanning trees of G. Then, for any set $F \subseteq E$, the sequence $\mathbb{P}_{T\sim\mu}[|F_T| = i], 0 \leq i \leq |F|$ corresponds to a sum of independent Bernoullis.*

Proof. First, by above discussion g_μ is real stable. Let $g_1(z)$ be a specialization of $g_\mu(z)$ where for each $e \notin F$, we let $z_e = 1$. In words, $g_1(z)$ is the generating polynomial of μ projected on F. So, g_1 is real stable. Now, let $g_2(t)$ be a univariate polynomial where we set all variables of $g_1(z)$ equal to t, so g_2 is real stable, i.e., it is real rooted. Let $a_0, \dots, a_{|F|}$ be the coefficients of $g_2(t)$. Since g_2 is real rooted by Lemma 1.3 (in background notes) there is a set of $|F|$ independent Bernoulli random variables $B_1, \dots, B_{|F|}$ where for any $0 \leq j \leq |F|$,

$$\mathbb{P}\left[B_1 + \cdots + B_{|F|} = j\right] = a_j.$$

The lemma follows from the fact that $a_i = \mathbb{P}_{T\sim\mu}[|F_T| = i]$. □

References

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