Summer School on Polynomial Paradigm in Algorithms **Summer 2020**

Problem Set 4

1) Recall that a polynomial $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ is log-concave if for any $x, y \in \mathbb{R}^n_{\geq 0}$ and any $0 < \alpha < 1$,

$$p(\alpha x + (1-\alpha)y) \geq p(x)^\alpha \cdot p(y)^{1-\alpha}.$$

For $A \in \mathbb{R}^{n \times m}_{\geq 0}$ and $b \in \mathbb{R}^n_{\geq 0}$, let $T : \mathbb{R}^m \to \mathbb{R}^n$ defined as $y \mapsto Ay + b$. For a log-concave polynomial $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$, prove that $p(T(y_1, \dots, y_m))$ has non-negative coefficients and is log-concave.

2) Prove the basis generating polynomial of any matroid with at most 5 elements is real stable.

3) Let $p \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]$ be a homogeneous multilinear log-concave polynomial. Prove that p is completely log-concave.

4) $g_\mu = \frac{1}{\#\text{Bases}} \sum_{B:\text{base}} z^B$ be generating polynomial of the uniform distribution over the bases of a given matroid $M = ([n], \mathcal{I})$.

   a) Show that
   $$\nabla^2 g_\mu(\mathbf{1}) \preceq (\nabla g_\mu(\mathbf{1}))(\nabla g_\mu(\mathbf{1}))^\intercal.$$

   b) Show that for any $1 \leq i < j \leq n$,
   $$2\mathbb{P}\left[i\right]\mathbb{P}\left[j\right] \geq \mathbb{P}\left[i, j\right].$$

   In other words, although uniform distribution over bases of a matroid is not necessarily negatively correlated, it is almost negative correlated (up to a factor 2).