Summer School on Polynomial Paradigm in Algorithms **Summer 2020**

Problem Set 3

1. Let x be a feasible solution of Held-Karp LP relaxation such that for $e_0 = \{u_0, v_0\}$, $x_{e_0} = 1$. Let $G = (V, E_0)$ be the support of x where $E_0 = E \cup \{e_0\}$. Let μ be a random spanning tree distribution such that for any $e \in E$, $\mathbb{P}_{T\sim\mu}[e] = x_e$. Let $A \subseteq V$ such that $u_0, v_0 \notin A$ and $x(\delta(A)) \leq 2(1+\epsilon)$. Show that
$$\mathbb{P}[|T \cap E(A)| = |A| - 1] \geq 1 - \epsilon.$$

2. Given a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$, the *homogenization* of p is defined as $p_H \in \mathbb{R}[z_1, \ldots, z_{n+1}]$ where
$$p_H = z_{n+1}^{\deg p} p(z_1/z_{n+1}, \ldots, z_n/z_{n+1}).$$
For example, if $p = z_1^2 z_2 - z_3$, then $p_H = z_1^2 z_2 - z_3 z_4^2$. Borcea and Brändén and Liggett proved the following theorem:

   Theorem 3.1. *If $p \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$ is real stable, then so is p_H.*

   For example, since $z_1 z_2 + z_2$ is real stable with non-negative coefficient, $z_1 z_2 + z_2 z_3$ is also real stable.

   a) Show that p having non-negative coefficient is a necessary condition in the above theorem.

   b) Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a strongly Rayleigh distribution of degree d. For an integer $1 \leq k \leq d$, the *truncation of μ to k*, μ_k is defined as follows:
$$\mu_k(S) = \begin{cases} \frac{\mu(S)}{\sum_{T:|T|=k} \mu(T)} & \text{if } |S| = k, \\ 0 & \text{otherwise.} \end{cases}$$
   Show that if μ is strongly Rayleigh, then so is μ_k for any integer $k \geq 1$ (as long as μ_k is well-defined).

3. We say a function $f : 2^{[n]} \to \mathbb{R}$ is *increasing* if for any $A, B \subseteq [n]$ such that $A \subseteq B$, $f(A) \leq f(B)$. We say a distribution $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is *stochastically dominated* by $\nu : 2^{[n]} \to \mathbb{R}_{\geq 0}$, $\mu \preceq \nu$, if for any increasing function $f : 2^{[n]} \to \mathbb{R}$, $\mathbb{E}_\mu[f] \leq \mathbb{E}_\nu[f]$. Borcea, Brändén and Liggett showed that for any strongly Rayleigh distribution μ and integer k,
$$\mu_k \preceq \mu_{k+1},$$
as long as μ_k, μ_{k+1} are well-defined. Let $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a strongly Rayleigh distribution. For disjoint sets $A, B \subseteq [n]$, let $T_A := |A \cap T|, T_B := |B \cap T|$ for $T \sim \mu$.

   a) For non-negative integers n_A, n_B and $n := n_A + n_B$ show that
$$\mathbb{P}[T_A \leq n_A | T_A + T_B = n] \geq \mathbb{P}[T_A \leq n_A | T_A + T_B \geq n].$$

   b) Recall that in assignment 1 we showed that any SR distribution is negatively correlated. Indeed, they prove a stronger property for strongly Rayleigh distributions, namely, *negative association.* For any two increasing functions $f, g : 2^{[n]} \to \mathbb{R}_{\geq 0}$ that depend on disjoint sets of coordinates, $\mathbb{E}[fg] \leq \mathbb{E}[f]\,\mathbb{E}[g]$. Show that
$$\mathbb{P}[T_A \leq n_A | T_A + T_B = n] \geq \mathbb{P}[T_A \leq n_A]\,\mathbb{P}[T_B \geq n_B].$$

   c) Prove that
$$\mathbb{P}[T_A = n_A, T_B = n_B | T_A + T_B = n] \geq \frac{1}{n} \min\{\mathbb{P}[T_A \leq n_A]\,\mathbb{P}[T_B \geq n_B], \mathbb{P}[T_A \geq n_A]\,\mathbb{P}[T_B \leq n_B]\}.$$