Summer School on Polynomial Paradigm in Algorithms **Summer 2020**

Problem Set 2

Given two polynomials $p, q \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$ we define the inner product of p, q as follows:

$$\langle p, q\rangle = q(\partial z)p|_{z=0} = p(\partial z)q|_{z=0} = \sum_{\kappa \in \mathbb{Z}^n_{\geq 0}} c_p(\kappa)c_q(\kappa)\kappa!,$$

where $\kappa! = \prod_{i=1}^n \kappa_i!$.

In this exercise we will sketch the proof of the following theorem.

Theorem 2.1. *For any two real stable polynomials $p, q \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$,*

$$\sup_{\alpha \in \mathbb{R}^n_{\geq 0}} e^{-\alpha} f(p, q, \alpha) \leq \langle p, q\rangle \leq \sup_{\alpha \in \mathbb{R}^n_{\geq 0}} f(p, q, \alpha) \tag{2.1}$$

where

$$f(p, q, \alpha) := \inf_{y, z \in \mathbb{R}^n_{>0}} \frac{p(y)q(z)}{(yz/\alpha)^\alpha}.$$

Recall that $(yz/\alpha)^\alpha = \prod_{i=1}^n (y_i z_i/\alpha_i)^{\alpha_i}$. Furthermore, in the special case that p, q are multilinear we can improve $e^{-\alpha}$ to $(1-\alpha)^{1-\alpha}$.

To motivate, first try to show that $\sup_{\alpha \geq 0} f(p, q, \alpha)$ can be computed in polynomial. This theorem has several applications and it has lead to several further results in this area. In this exercise, we sketch the proof of the LHS, i.e., the hard direction of the above equation.

1) **Optional:** Prove that $\sup_{\alpha \in \mathbb{R}^n_{\geq 0}} f(p, q, \alpha)$ can be turned into a convex program. More precisely, a saddle point concave-convex problem, i.e., prove that $\log f$ is a concave function of α and its value is inf of a convex function. It can be shown that this program has a polynomial time algorithm using ellipsoid method.

2) The statement about all real stable polynomials can be reduced to multilinear ones by the polarization technique. Given a polynomial $p \in \mathbb{R}[z_1, \ldots, z_n]$, let $m \geq \max_i \deg_{z_i} p$. The polarization of p, $\pi_m(p) = \mathbb{R}[z_{1,1}, \ldots, z_{1,m}, \ldots, z_{n,m}]$ is the multilinear polynomial defined as follows: For any $1 \leq i \leq n$ and $1 \leq j \leq \deg_{z_i} p$ we substitute any occurrence of z_i^j with $\frac{1}{\binom{m}{j}} e_j(z_{i,1}, \ldots, z_{i,m})$. Note that this polynomial is *symmetric* under all clonings of z_i and if we specialize all $z_{i,j}$'s with z_i we get back the polynomial p. Borcea, Brändén and Liggett proved the following theorem:

   Theorem 2.2 ([BBL08])**.** *$p \in \mathbb{R}[z_1, \ldots, z_n]$ is stable iff $\pi_m(p)$ is stable for all $m \geq \max_i \deg_{z_i} p$.*

   Suppose that for any two multilinear real stable polynomials $p, q \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$, and any $\alpha \in \mathbb{R}^n_{\geq 0}$,

   $$(1-\alpha)^{1-\alpha} f(p, q, \alpha) \leq \langle p, q\rangle.$$

   Prove the left side of (2.1) for any polynomials p, q (which are not necessarily multilinear).

3) Prove that for any two multilinear polynomials $p \in \mathbb{R}[y_1, \ldots, y_n], q \in \mathbb{R}[z_1, \ldots, z_n]$,

   $$\langle p, q\rangle = \prod_{i=1}^n (1 + \partial_{y_i}\partial_{z_i}) pq|_{y=z=0}.$$

4) Unfortunately, $1 + \partial_{y_i}\partial_{z_i}$ is not a stability preserver operator. So, we cannot immediately invoke Gurvits' like technique. Instead, we need a stronger induction hypothesis. Say, a polynomial $r \in \mathbb{R}[y_1, \ldots, z_1, \ldots, z_n]$ is bi-stable if $r(y_1, \ldots, y_n, -z_1, \ldots, -z_n)$ is stable. Show that if $p, q \in \mathbb{R}_{\geq 0}[z_1, \ldots, z_n]$ are stable then $r = p \cdot q$ is bi-stable.

5) Show that if r is bi-stable so is $(1 + \partial_{y_i}\partial_{z_i})r|_{y_i=z_i=0}$.

6) Use induction to prove that for a multilinear bi-stable polynomial $p \in \mathbb{R}_{\geq 0}[y_1, \ldots, y_n, z_1, \ldots, z_n]$ and any $\alpha \in \mathbb{R}^n_{\geq 0}$,

$$\prod_{i=1}^{n}(1 + \partial_{y_i}\partial_{z_i})p|_{y=z=0} \geq (1-\alpha)^{1-\alpha} \inf_{y,z>0} \frac{p(y,z)}{(yz/\alpha)^{\alpha}}.$$

For this exercise, assume that the statement holds for $n = 1$.