Integrality Gap for Max-Cut / GW SDP.

$G(V, E)$. $V = \{1, 2, \dots, n\}$.

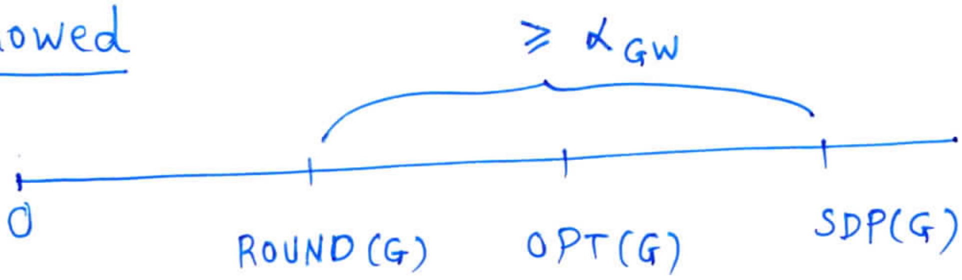
$$\mathrm{OPT}(G) = \max \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\forall i, \quad x_i \in \{-1, 1\}.$$

$$\mathrm{SDP}(G) = \max \sum_{(i,j)\in E} \frac{1 - v_i \cdot v_j}{2}$$

$$\forall i, \quad \|v_i\| = 1.$$

<u>We showed</u>



[Karloff, Feige Schechtman]

- $\exists \{G_k\}$ s.t. $\displaystyle\lim_{k\to\infty} \frac{\mathrm{ROUND}(G_k)}{\mathrm{SDP}(G_k)} = \alpha_{GW}$.
- $\exists \{G_k\}$ s.t. $\displaystyle\lim_{k\to\infty} \frac{\mathrm{OPT}(G_k)}{\mathrm{SDP}(G_k)} = \alpha_{GW}$.

i.e. the "integrality gap" of the SDP is α_{GW}.

<u>Def</u>.

Integrality Gap β (of specific <u>SDP</u> relax^n) $\displaystyle = \inf_G \frac{\mathrm{OPT}(G)}{\mathrm{SDP}(G)}$.

Usually if integrality gap is β then it is taken as evidence that the problem is hard to β-approximate (via the specific SDP; this is a weaker criterion of hardness than NP-hardness).

— x —

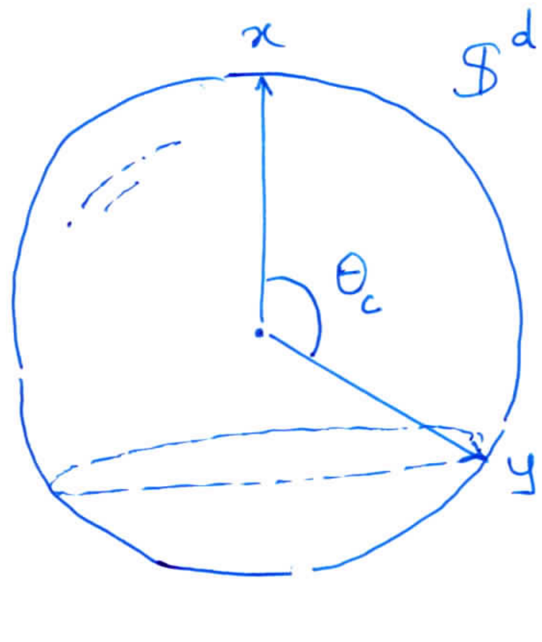
We'll construct a graph $G(V,E)$ s.t.

$$\frac{OPT(G)}{SDP(G)} \leq \alpha_{GW} + o(1).$$

[Description is imprecise; details omitted].

$V = \mathbb{S}^d$ $\qquad\qquad$ $\theta_c \approx 117°$.

$E = \{ (x,y) \mid \text{Angle}(x,y) = \theta_c \}$



Claim $\quad SDP(G) \geq \frac{1-\cos\theta_c}{2}.$

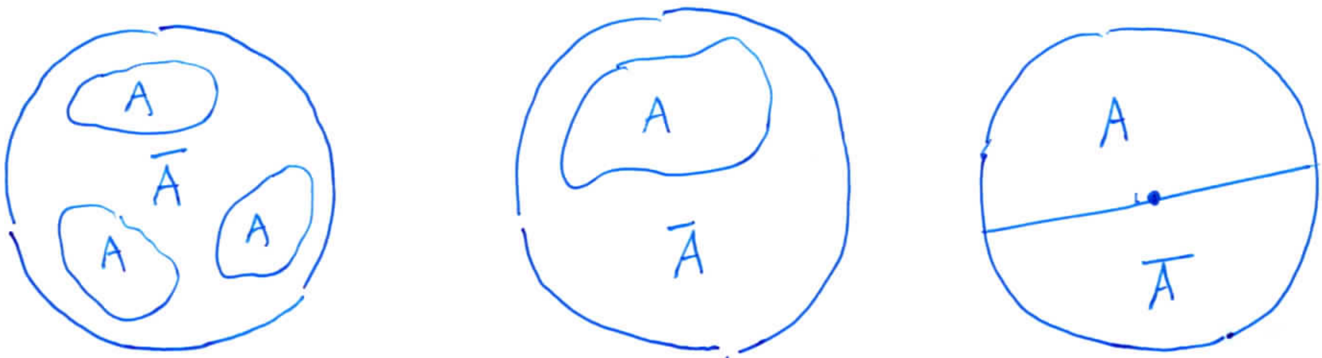
Proof Consider a vector solution

$$x \longrightarrow v_x \quad (\text{the vector } x \text{ itself}).$$

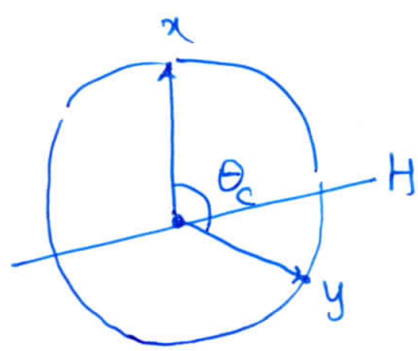
$$SDP(G) \geq \mathop{\mathbb{E}}_{(x,y)\in E}\left[\frac{1 - v_x \cdot v_y}{2}\right] = \frac{1-\cos\theta_c}{2}.$$

Claim $\quad OPT(G) \leq \frac{\theta_c}{\pi}.$

Proof. $\quad$ cut: $\mathbb{S}^d = A \cup \bar{A}.$



Theorem Among all cuts, max-cut is achieved by a half-sphere.



$$Cut_H = \Pr_{(x,y)\in E}\left[(x,y) \text{ cut by } H\right]$$

$$= \Pr_{H'}\left[(x,y) \text{ cut by } H'\right]$$

$$= \theta_c / \pi.$$

$\therefore \quad OPT(G) = \dfrac{\theta_c}{\pi}.$

$$\therefore \quad \frac{OPT(G)}{SDP(G)} \leq \frac{\theta_c / \pi}{\frac{1-\cos\theta_c}{2}} = \alpha_{GW.} + \underbrace{o(1)}$$

Details : - Discretization

- $d \to \infty$

- Actual geometric theorem is a bit different.

Hypercontractivity & Noisy Hypercube

Def For $f : \{-1,1\}^n \to \mathbb{R}$ let $1 \le p < \infty$

$$\|f\|_p = \left(\mathbb{E}_x \left[|f(x)|^p \right] \right)^{1/p}.$$

Fact $\|f\|_p$ is increasing in p, i.e. if $p \le q$ then $\|f\|_p \le \|f\|_q$.

Proof

$$\mathbb{E}_x \left[|f(x)|^p \right]^{1/p} \le \mathbb{E}_x \left[|f(x)|^q \right]^{1/q}.$$

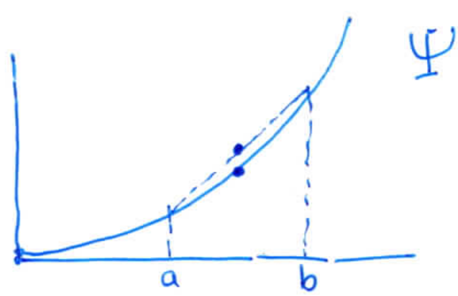
Let $g(x) = |f(x)|^p \geq 0$. Same as,

$$\mathbb{E}_x \left[g(x) \right]^{q/p} \le \mathbb{E}_x \left[g(x)^{q/p} \right]$$

Same as

$$\Psi \left(\mathbb{E}_x \left[g(x) \right] \right) \le \mathbb{E}_x \left[\Psi(g(x)) \right]$$

where $\Psi(\theta) = \theta^{q/p}$ is convex since $q/p \geq 1$.



$$\Psi\left(\frac{a+b}{2}\right) \le \frac{\Psi(a) + \Psi(b)}{2}.$$

$\blacksquare$

Recall Beckner operator

$$T_{1-\gamma}(f) = \sum_S \hat{f}(S)\,(1-\gamma)^{|S|}\,\chi_S.$$

Equivalently $$T_{1-\gamma}(f)(x) = \mathop{\mathbb{E}}_{y \underset{1-\gamma}{\sim} x}\left[f(x)\right].$$

Fact $$\|T_{1-\gamma} f\|_p \leq \|f\|_p. \qquad 1 \leq p.$$

Proof
$$\|T_{1-\gamma} f\|_p^p = \mathop{\mathbb{E}}_x\left[\,|T_{1-\gamma} f\,(x)|^p\right]$$

$$= \mathop{\mathbb{E}}_x\left[\,\left|\mathop{\mathbb{E}}_{y \underset{1-\gamma}{\sim} x}\left[f(y)\right]\right|^p\right]$$

$$\leq \mathop{\mathbb{E}}_x\left[\mathop{\mathbb{E}}_{y \underset{1-\gamma}{\sim} x}\left[|f(y)|^p\right]\right] \quad \because \text{convexity}$$

$$= \mathop{\mathbb{E}}_y\left[|f(y)|^p\right]$$

$$= \|f\|_p^p. \qquad \blacksquare$$

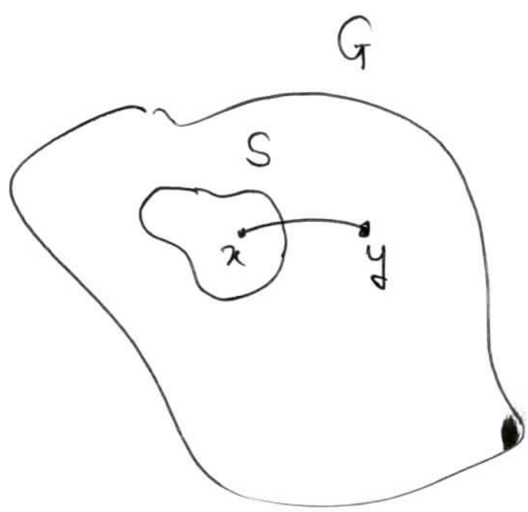
$\boxed{T_{1-\gamma} \text{ is Contractive.}}$

Theorem (Bonami-Beckner, Hyper contractivity).

$$\| T_{1-\gamma} f \|_2 \leq \| f \|_{1+(1-\gamma)^2}.$$

Expansion of small sets in Noisy Hypercube

- $G(\{-1,1\}^n, E)$ weighted.
- For fixed $x \in \{-1,1\}^n$, a "random edge" (x,y) going out of x is sampled by $y \sim_{1-\gamma} x$.



Def Let $S \subseteq \{-1,1\}^n$.

$|S| = \delta \cdot 2^n$.

Expansion of S

$$\Phi(S) = \Pr_{\substack{x \in S \\ y \sim_{1-\gamma} x}} [y \notin S].$$

<u>Fact</u> $\Phi(S) \geqslant 1 - \delta^{\gamma/2}$ $\quad (\to 1 \text{ as } \delta \to 0)$.

<u>Proof</u>

$$1 - \Phi(S) = \Pr_{\substack{x \in S \\ y \sim_{1-\gamma} x}} [y \in S]$$

$$= \frac{\Pr_{x,\, y \sim_{1-\gamma} x} [x \in S,\, y \in S]}{\Pr_x [x \in S]}$$

$$= \frac{1}{\delta} \cdot \mathbb{E}_{x,\, y \sim_{1-\gamma} x} [f(x) f(y)]$$

$f = \mathbb{1}_S$

$f(x) = \begin{cases} 1 & x \in S \\ 0 & \text{oth} \end{cases}$

$$= \frac{1}{\delta} \sum_S \hat{f}(S)^2 (1-\gamma)^{|S|}$$

$$= \frac{1}{\delta} \sum_S \left(\hat{f}(S) \sqrt{1-\gamma}^{\,|S|} \right)^2$$

$$= \frac{1}{\delta} \sum_S \left(\widehat{T_{\sqrt{1-\gamma}} f} (S) \right)^2$$

$$= \frac{1}{\delta} \left\| T_{\sqrt{1-\gamma}} f \right\|_2^2$$

$$\leq \frac{1}{\delta} \left\| f \right\|_{1+1-\gamma}^2$$

$$= \frac{1}{\delta} \left(\mathop{\mathbb{E}}_{x} \left[|f(x)|^{2-\gamma} \right]^{\frac{1}{2-\gamma}} \right)^2$$

$$= \frac{1}{\delta} \cdot \delta^{\frac{2}{2-\gamma}}$$

$$\frac{2}{2-\gamma} \geq 1 + \frac{\gamma}{2}$$

$$\leq \frac{1}{\delta} \delta^{1+\frac{\gamma}{2}}$$

$$= \delta^{\frac{\gamma}{2}}.$$

∎

" small sets in noisy hypercube expand nearly fully. "