Unique Games Conjecture $\Rightarrow$ GW algorithm for Max-cut is optimal.

Unique Label Cover is hard.

Dictatorship test [Majority is Unstablest under $\rho < 0$]

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Hardness of Max-cut

UGC Given unique label cover $\mathcal{L}$, NP-hard to tell if $OPT(\mathcal{L}) \geq 1-\eta$ or $OPT(\mathcal{L}) \leq \delta$.

Unique Label Cover $\longrightarrow$ max-cut

$$\mathcal{L} \rightsquigarrow G(V,E)$$

$$OPT(\mathcal{L}) \geq 1-\eta \Rightarrow OPT(G) \geq \frac{1-\rho}{2} - \varepsilon$$

$$OPT(\mathcal{L}) \leq \delta \Rightarrow OPT(G) \leq \frac{1}{\pi}\cos^{-1}\rho + \varepsilon.$$

— x —

$\rho = \cos\theta_c \in (-1,0)$, $\theta_c \approx 117°$.

Dictatorship Test

$\rho = \cos\theta_c < 0.$

Given $f : \{-1,1\}^n \to \{-1,1\}$

- Pick $x \in \{-1,1\}^n$ u.a.r., $y \sim_\rho x$.
- Accept iff $f(x) \neq f(y)$.

(comp.) If $f = \text{Dict}_{i_0}$, $\Pr[\text{Accept}] = \frac{1-\rho}{2}$.

(Sound.) If $\Pr[\text{Accept}] \geq \frac{1}{\pi}\cos^{-1}\rho + \varepsilon$ then

$\exists\, j \in \{1, \dots, n\}$ s.t. $\text{Inf}_j^{\leq k}(f) \geq \delta$.

— x —

Note - $\Pr[\text{Accept}] = 1 - \text{Stab}_\rho(f)$.

(Comp). $\Pr[\text{Accept}] = 1 - \text{Stab}_\rho(\text{Dict}) = 1 - \frac{1+\rho}{2}$.

(Sound). Contrapositive of Theorem 3.

$$\neg \quad \text{Stab}_\rho(f) \geq 1 - \frac{1}{\pi}\cos^{-1}\rho - \varepsilon$$

$$\text{Stab}_\rho(f) \leq 1 - \frac{1}{\pi}\cos^{-1}\rho - \varepsilon$$

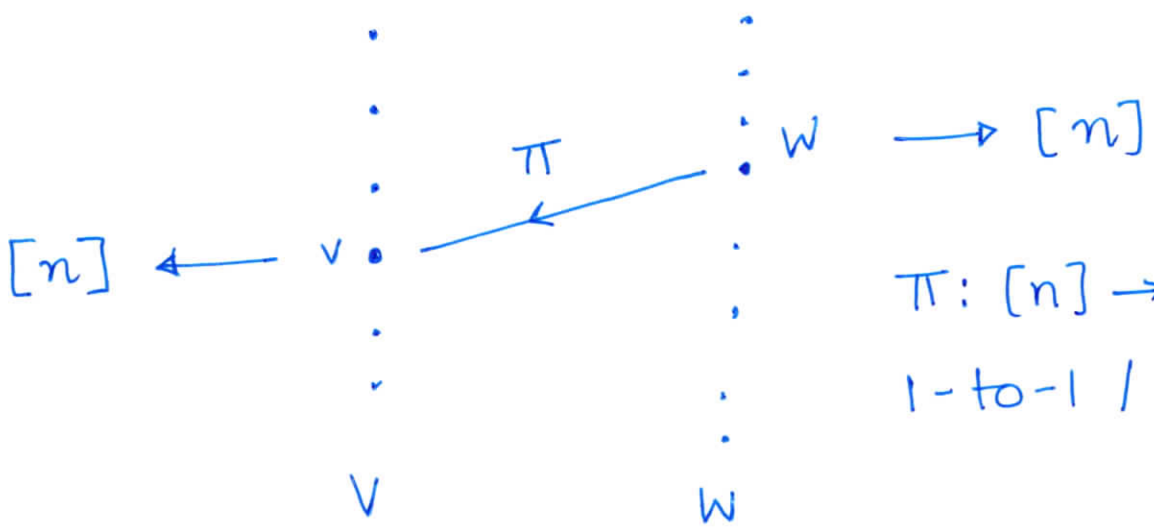
$$1 - \text{Stab}_\rho(f) \geq \frac{1}{\pi}\cos^{-1}\rho + \varepsilon.$$

Unique Games Conjecture

Def: A Label Cover instance

$$\mathcal{L}\left(G(V, W, E),\ [n],\ [m],\ \{\pi_{vw} : [m] \to [n]\}_{(v,w) \in E} \right)$$

is unique if
- $m = n$.
- π_{vw} are bijections.



$\pi : [n] \to [n]$,
1-to-1 / bijection.

UGC $\forall$ constant $\eta, \delta > 0$

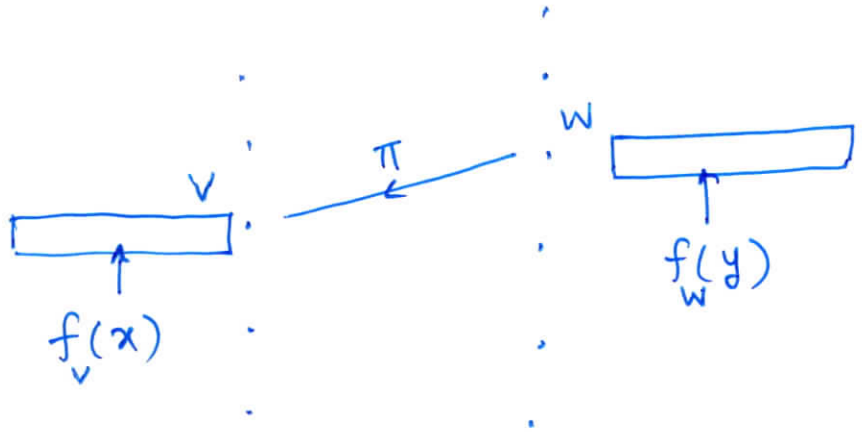
$\exists$ constant integer n s.t.

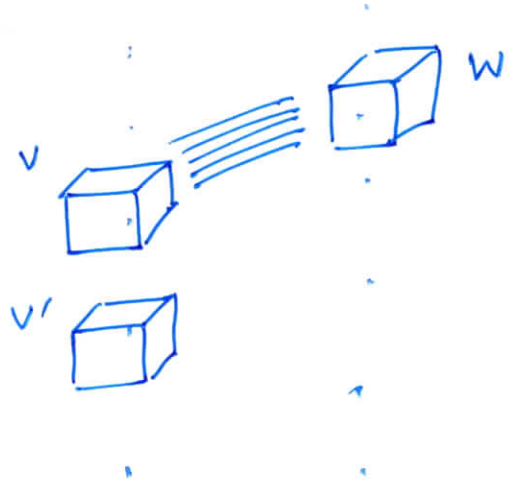
given Unique Label Cover instance $\mathcal{L}$

it is NP-hard to distinguish

if $OPT(\mathcal{L}) \geq 1-\eta$ or $OPT(\mathcal{L}) \leq \delta$.

First Attempt (Long codes)



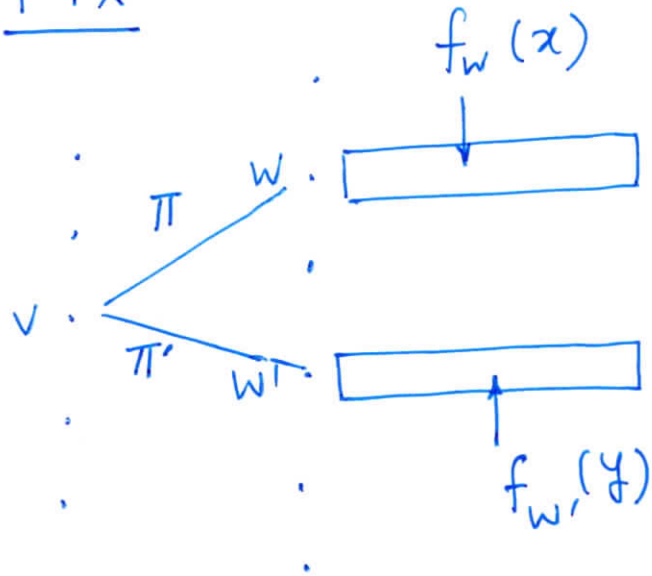


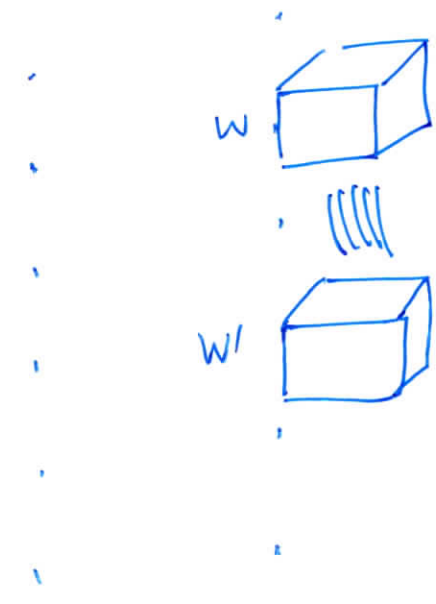
$f_v : \{-1,1\}^n \to \{-1,1\}$ $\qquad$ $f_w : \{-1,1\}^n \to \{-1,1\}$

$\mathcal{L} \rightsquigarrow$ Graph.

Bipartite :(

Fix





$\pi \circ \pi'$ is also bijection.

virtual

Non-bipartite :)

For simplicity, assume all bijections $\pi : [n] \to [n]$ are identity. Do all permutations mentally.

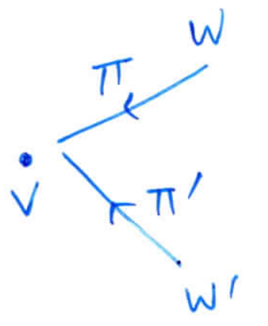
PCP Given $\mathcal{L}(G(V, W, E), [n], \{\pi_{vw}\})$

Given $f_w : \{-1,1\}^n \to \{-1,1\}$ $\forall w \in W$.

(supposed long code of $\ell(w)$).

- Pick $v \in V$, and neighbors $w, w' \in W$.
- Pick $x \in \{-1,1\}^n$, $y \sim_\rho x$.
- Accept iff $f_w(x) \neq f_{w'}(y)$.

Note Actually



- $f_w(x \circ \pi) \neq f_{w'}(y \circ \pi')$
- $x \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$.

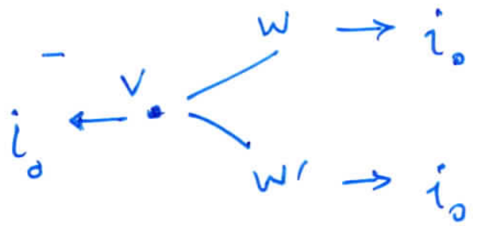
$\pi(\ell(w)) = \ell(v)$

$\pi'(\ell(w')) = \ell(v)$

Completeness

Suppose $OPT(\mathcal{L}) \geq 1-\eta$. Let ℓ be labeling.

- Edges (v,w), (v,w') both satisfied w.p. $1-2\eta$.

- 

$f_w = \mathrm{Dict}_{i_0}$

$f_{w'} = \mathrm{Dict}_{i_0}$

$\therefore \Pr[\text{Accept}] = \dfrac{1-\rho}{2}$

$\therefore \Pr[\text{Overall Accept}] \geq (1-2\eta)\dfrac{1-\rho}{2} \geq \dfrac{1-\rho}{2} - \eta.$

Soundness

Idea If $\Pr[\text{Accept}] \geq \frac{1}{\pi}\cos^{-1}\rho + \varepsilon$

then $\exists j$ $\quad \mathrm{Inf}_j^{\leq k}(f_w) \geq \delta.$

$\mathrm{Inf}_j^{\leq k}(f_{w'}) \geq \delta.$

Use label j to w, w'.

Randomized, bounded list, what about decoding v ?

Soundness Analysis

$$\Pr[\text{Overall Accept}] = \mathop{\mathbb{E}}_{\substack{v,\, w,\, w' \\ x,\, y}}\left[\frac{1 - f_w(x)\, f_{w'}(y)}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{v, x, y}\left[\mathop{\mathbb{E}}_{w, w'}\left[f_w(x)\, f_{w'}(y)\right]\right]$$

$$= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{v, x, y}\left[\mathop{\mathbb{E}}_{w}\left[f_w(x)\right] \cdot \mathop{\mathbb{E}}_{w'}\left[f_{w'}(y)\right]\right]$$

$$= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{v, x, y}\left[f_v(x)\, f_v(y)\right]$$

$$= \mathop{\mathbb{E}}_{v}\left[1 - \mathrm{Stab}_\rho(f_v)\right] \quad \text{where}$$

$$f_v(x) \stackrel{\text{def}}{=} \mathop{\mathbb{E}}_{w \sim v}\left[f_w(x)\right] \qquad :).$$

f_v is $[-1, 1]$-valued.

Now suppose

$$\Pr[\text{overall Accept}] \geq \frac{1}{\pi}\cos^{-1}\rho + \varepsilon.$$

By averaging argument, for $\geq \frac{\varepsilon}{2}$ fraction of v (call such v "good", fix a "good" v),

$$1 - \mathrm{Stab}_\rho(f_v) \geq \frac{1}{\pi}\cos^{-1}\rho + \frac{\varepsilon}{2}.$$

$\therefore$ f_v must have a co-ordinate i_0

s.t. $\mathrm{Inf}_{i_0}^{\leq k}(f_v) \geq \delta$.

<u>(Informal)</u> i_0 must also have degree-k influence $\geq \frac{\delta}{2}$ for "many" $w \sim v$.

<u>Proof</u>

$$\delta \leq \mathrm{Inf}_{i_0}^{\leq k}(f_v)$$

$$= \sum_{\substack{i_0 \in S \\ |S| \leq k}} \hat{f}_v(S)^2 \qquad S \subseteq [n].$$

$$= \sum_{\substack{i_0 \in S \\ |S| \leq k}} \mathop{\mathbb{E}}_{w}\left[\hat{f}_w(S)\right]^2 \qquad \because f_v = \mathbb{E}[f_w]$$

$$\leq \sum_{\substack{i_0 \in S \\ |S| \leq k}} \mathop{\mathbb{E}}_{w}\left[\hat{f}_w(S)^2\right] \qquad \text{Cauchy-Shwartz!}$$

$$= \mathop{\mathbb{E}}_{w}\left[\sum_{\substack{i_0 \in S \\ |S| \leq k}} \hat{f}_w(S)^2\right]$$

$$= \mathop{\mathbb{E}}_{w}\left[\mathrm{Inf}_{i_0}^{\leq k}(f_w)\right].$$

$\therefore$ By averaging argument,

for $\geq \frac{\delta}{2}$ fraction of $w \sim v$,

$$\mathrm{Inf}_{i_0}^{\leq k}[f_w] \geq \frac{\delta}{2}.$$

Labeling (randomized)

Randomly from $\{ i \mid \mathrm{Inf}_i^{\le k}(f_w) \geq \frac{\delta}{2} \}$

- For $w \in W$, assign randomly from

  $$\mathrm{Cand}(w) = \{ i \mid \mathrm{Inf}_i^{\le k}(f_w) \geq \frac{\delta}{2} \}.$$

  $$|\mathrm{Cand}(w)| \le k/(\tfrac{\delta}{2}) = 2k/\delta.$$

- For $v \in V$,
  - Pick $w' \sim v$ at random.
  - Assign randomly from $\mathrm{Cand}(w')$.

We concluded that

For $\frac{\varepsilon}{2}$ fraction of v, $\exists\, i_0 \in [n]$

For $\frac{\delta}{2}$ fraction of $w \sim v$,

$i_0 \in \mathrm{Cand}(w)$.

$\therefore$ In the (randomized) labeling, fraction of edges (v,w) satisfied is

$$\frac{\varepsilon}{2} \cdot \frac{\delta}{2} \cdot \frac{\delta}{2} \cdot \underbrace{1/(2k/\delta)^2}.$$

(i.e.
$v \to i_0$
$w \to i_0$
$w' \to i_0$

$|\mathrm{Cand}(w')|$.
$|\mathrm{Cand}(w)|$.).

$\therefore$ Provided $\mathrm{OPT}(\mathcal{L})$ was small enough to begin with,

$$\Pr[\text{Overall Accept}] \leq \frac{1}{\pi} \cos^{-1} \rho + \varepsilon.$$