Max-Cut

- GW-algorithm
- α_{GW}-approx.

$$\alpha_{GW} = \frac{\theta_c/\pi}{(1-\cos\theta_c)/2}$$

≈ 0.878 at

$\theta_c \approx 117°$.

- UGC $\Rightarrow$ GW is optimal.
- $\boxed{\text{Majority is Stablest.}}$

<u>High level</u> UGC: Given "unique" game/label cover $\mathcal{L}$, it is NP-hard to tell

$OPT(\mathcal{L}) \geq 1-\varepsilon$ <u>or</u> $OPT(\mathcal{L}) \leq \delta$.

$$\text{Unique Games} \xrightarrow{\text{Reduction}} \text{MAX-CUT}$$

$$\mathcal{L} \rightsquigarrow G(V,E)$$

$$OPT(\mathcal{L}) \geq 1-\varepsilon \implies OPT(G) \geq \frac{1-\cos\theta_c}{2} - \eta$$

$$OPT(\mathcal{L}) \leq \delta \implies OPT(G) \leq \frac{\theta_c}{\pi} + \eta$$

$\blacksquare$

Noise Stability

- Fix $\rho \in (0,1)$ (think of $\rho = 1-2\varepsilon$, ε tiny).
- Fix $x \in \{-1,1\}^n$ for now.

Def $y \in \{-1,1\}^n$ is a random input ρ-correlated with x (denoted $y \sim_\rho x$) if

$$y_i = \begin{cases} x_i & \text{with prob. } \rho \\ -1, +1 & \text{at random with prob. } 1-\rho. \end{cases}$$

Equivalently ("noise")

$$y_i = \begin{cases} x_i & \text{with prob. } 1-\varepsilon \\ -x_i & \text{with prob. } \varepsilon \end{cases} \qquad \boxed{\rho = 1-2\varepsilon}$$

Note

- $\underbrace{\mathbb{E}[x_i y_i]}_{\text{correlation}} = \rho \cdot x_i^2 + (1-\rho)\,\mathbb{E}[\pm 1] = \rho.$

- If $x \in \{-1,1\}^n$ is uniformly random, then so is $y \sim_\rho x$. — Not^n: $y = x\mu$, $\mathbb{E}[\mu_i] = \rho$.

Def For $f:\{-1,1\}^n \to \{-1,1\}$, its noise-stability

$$\mathrm{Stab}_\rho(f) = \Pr_{x,\, y \sim_\rho x}\left[f(x) = f(y) \right].$$

We can express $\mathrm{Stab}_\rho(f)$ in terms of Fourier coeffs.

$$\mathrm{Stab}_\rho(f) = \Pr_{x,\, y \sim_\rho x}\left[f(x) = f(y) \right] \qquad y = x\mu, \quad \mathbb{E}[\mu_i] = \rho.$$

$$= \mathbb{E}_{x,\, y \sim_\rho x}\left[\frac{1 + f(x) f(y)}{2} \right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{S,T \subseteq [n]} \hat{f}(S) \hat{f}(T)\, \mathbb{E}\left[\chi_S(x)\, \chi_T(y = x\mu) \right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{S,T} \hat{f}(S) \hat{f}(T)\, \mathbb{E}_x\left[\chi_S(x)\, \chi_T(x) \right] \mathbb{E}_\mu\left[\chi_T(\mu) \right]$$

$$\overset{S=T}{=} \frac{1}{2} + \frac{1}{2} \sum_S \hat{f}(S)^2 \rho^{|S|}.$$

$$\boxed{\therefore \quad \mathrm{Stab}_\rho(f) = \frac{1}{2} + \frac{1}{2} \sum_S \hat{f}(S)^2 \rho^{|S|}.}$$

Note. This formula makes sense even if $f: \{-1,1\}^n \to [-1,1]$ is real-valued.

Goal (for now)

Which function $f : \{-1,1\}^n \to \{-1,1\}$, $\mathbb{E}[f] = 0$,

maximize $\mathrm{Stab}_\rho(f)$?

Answer : Dictatorship, $f = x_S$, $|S| = 1$.

Proof $\hat{f}(\phi) = \mathbb{E}_x[f] = 0$.

$$\therefore \mathrm{Stab}_\rho(f) = \frac{1}{2} + \frac{1}{2}\sum_S \hat{f}(S)^2 \rho^{|S|}$$

$$\leq \frac{1}{2} + \frac{1}{2}\sum_S \hat{f}(S)^2 \cdot \rho \qquad \because \hat{f}(\phi) = 0.$$

$$= \frac{1}{2} + \frac{1}{2}\rho,$$

and equality iff f is dictatorship.

Observe $\rho = 1 - 2\varepsilon$.

$\mathrm{Stab}_\rho(\mathrm{DICT}) = 1 - \varepsilon$

$\mathrm{Stab}_\rho(k\text{-JUNTA}) \geq 1 - k\varepsilon$

k-JUNTA is a function that depends on $\leq k$ coordinates

$\mathrm{Stab}_\rho(\mathrm{PARITY}) = \frac{1}{2} + \frac{1}{2}(1-2\varepsilon)^n$

$= \frac{1}{2} + o(1)$

$\mathrm{PARITY} = \prod_{i=1}^{n} x_i$.

Question What is the highest stability for functions that
- do not depend on few variables
$\equiv$ "far" from being dictator.

$\frac{1}{2} + o(1)$? Not quite!

Def

$$\text{Majority}(x_1, \ldots, x_n) = \begin{cases} +1 & \text{if } \sum_{i=1}^{n} x_i \geq 0 \\ -1 & \text{if } \sum_{i=1}^{n} x_i < 0. \end{cases}$$

Fact

- $\text{Stab}_\rho(\text{Majority}) = 1 - \theta(\sqrt{\varepsilon})$ $\quad \rho = 1 - 2\varepsilon$.
- $\lim_{n \to \infty} \text{Stab}_\rho(\text{Majority}_n) = 1 - \frac{1}{\pi} \cos^{-1} \rho. > \frac{1}{2}.$

intuitive reasoning

- For typical x, $\sum_{i=1}^{n} x_i \in [-\sqrt{n}, \sqrt{n}]$.
- $\sqrt{\varepsilon}$ fraction of x, $\sum_{i=1}^{n} x_i \in [-\sqrt{\varepsilon n}, \sqrt{\varepsilon n}]$
- For these x, flipping εn coordinates changes sign with constant probability. ▨

Influences of function $f:\{-1,1\}^n \to \{-1,1\}$.

- To formalize notion that f does not depend on few variables.

Def For $1 \leq i \leq n$,

$$\mathrm{Inf}_i(f) = \Pr_x \left[f(x_1, \cdots, x_{i-1}, \underline{x_i}, x_{i+1}, \cdots x_n) \neq f(x_1, \cdots, x_{i-1}, \underline{-x_i}, x_{i+1}, \cdots, x_n) \right].$$

Equiv. $\mathrm{Inf}_i(f) = \Pr_x \left[f(x) \neq f(x e_i) \right]$ $\quad e_i = (1, 1, \cdots, \underset{\uparrow \, i^{th}}{-1}, 1, \cdots 1)$.

Fact

$$\mathrm{Inf}_i(f) = \sum_{i \in S} \hat{f}(S)^2. \quad \text{——— } \#1$$

Proof

$$\mathrm{Inf}_i(f) = \mathop{\mathbb{E}}_x \left[\frac{1 - f(x) f(x e_i)}{2} \right]$$

$$= \frac{1}{2} - \frac{1}{2} \sum_S \hat{f}(S)^2 \chi_S(e_i)$$

$$= \frac{1}{2} \sum_S \hat{f}(S)^2 (1 - \chi_S(e_i)) \qquad \because \sum_S \hat{f}(S)^2 = 1$$

$$= \sum_{i \in S} \hat{f}(S)^2.$$

Note: #1 makes sense for $f: \{-1,1\}^n \to [-1,1]$.

Examples

PARITY $\prod_{i=1}^{n} x_i$ $\quad$ $Inf_i(f) = 1 \quad \forall\ 1 \le i \le n.$

DICTATOR x_{i_0} $\quad$ $Inf_{i_0}(f) = 1$, $Inf_i(f) = 0$ for $i \neq i_0$.

Majority $\sum_{i=1}^{n} x_i \overset{?}{\geq} 0$ $\quad$ $Inf_i(f) = \Theta\left(\frac{1}{\sqrt{n}}\right) \quad \forall\ 1 \le i \le n.$

Reasoning $f(x) \neq f(x e_i)$ iff x is exactly in the "middle layer" of hypercube (i.e. exactly half $+1$, half -1 (if n even)), which constitutes $\Theta\left(\frac{1}{\sqrt{n}}\right)$ fraction of hypercube.

Theorem (Informal)

Among all functions $f: \{-1,1\}^n \to [-1,1]$, balanced ($\mathbb{E}[f] = 0$), with all influences low, Majority is most noise stable.

Theorem (1) [MAJORITY IS STABLEST, MOO'05]

Fix $\rho \in (0,1)$. For every $\varepsilon > 0$, there is $\delta = \delta(\rho, \varepsilon)$ s.t. for

- $f : \{-1,1\}^n \to [-1,1]$, $\mathbb{E}[f] = 0$,
- $\mathrm{Inf}_i(f) \leq \delta \quad \forall\, 1 \leq i \leq n$, we have

$$\mathrm{stab}_\rho(f) \leq \underbrace{1 - \frac{1}{\pi}\cos^{-1}\rho}_{\lim_{n\to\infty} \mathrm{stab}_\rho(\mathrm{majority}_n)} + \varepsilon .$$

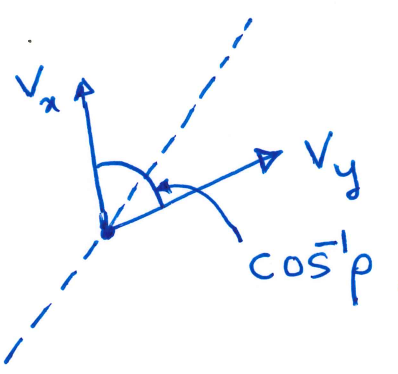
—— x ——

- Why is $\mathrm{Stab}_\rho(\mathrm{Majority}_n) \simeq 1 - \frac{1}{\pi}\cos^{-1}\rho$?
- Choose $x,\ y \sim_\rho x \in \{-1,1\}^n$.

$$V_x = \left(\frac{x_1}{\sqrt{n}}, \frac{x_2}{\sqrt{n}}, \ldots, \frac{x_n}{\sqrt{n}}\right) \qquad \|V_x\| = 1$$

$$V_y = \left(\frac{y_1}{\sqrt{n}}, \frac{y_2}{\sqrt{n}}, \ldots, \frac{y_n}{\sqrt{n}}\right) \qquad \|V_y\| = 1$$

$$V_x \cdot V_y \simeq \mathbb{E}[x_i y_i] = \rho .$$



$$\mathrm{sign}\left(\sum_{i=1}^{n} z_i\right) = \mathrm{Majority}$$