Proof of 2-to-2 Games Theorem

3 Lin $\longrightarrow$ 2-to-2 Games

$\uparrow$

(G) Linearity Testing Thm

$\uparrow$ ✓

(G) Expansion Thm ✓

- 3 Lin
- Parallel Repetition of smooth game

- Grassmann Code
- Lin. Testing Thm

compose

+

2-to-2 Games Theorem

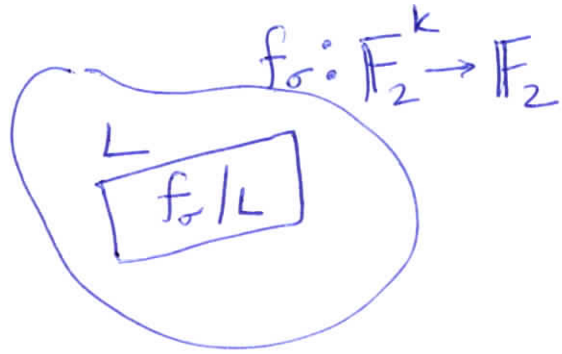
Grassmann Code

- To encode a string $\sigma \in \mathbb{F}_2^k$

or equivalently linear f^n $f_\sigma : \mathbb{F}_2^k \to \mathbb{F}_2$

$$f_\sigma(z) = \sum_{i=1}^{k} \sigma_i z_i$$

Write down table



$$F[L] = f_\sigma |_L$$

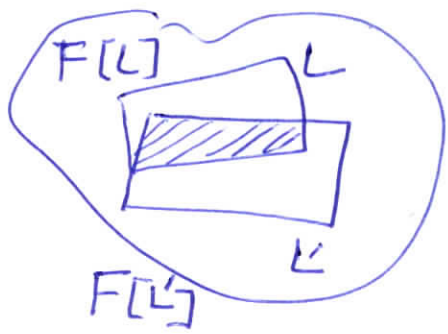
$$\forall L \subseteq \mathbb{F}_2^k, \ \dim(L) = \ell.$$

— x —

Grassmann Linearity Test

- Pick (L, L') s.t. $\dim(L \cap L') = \ell - 1$.
- Accept iff

$$F[L]|_{L \cap L'} = F[L']|_{L \cap L'}.$$



Given instance of 3Lin

Vars. $x_1, x_2, \ldots, x_n$

Eqns. E_1

$\vdots$

$E_i \quad x_{i_1} + x_{i_2} + x_{i_3} = 1$ over $\mathbb{F}_2$.

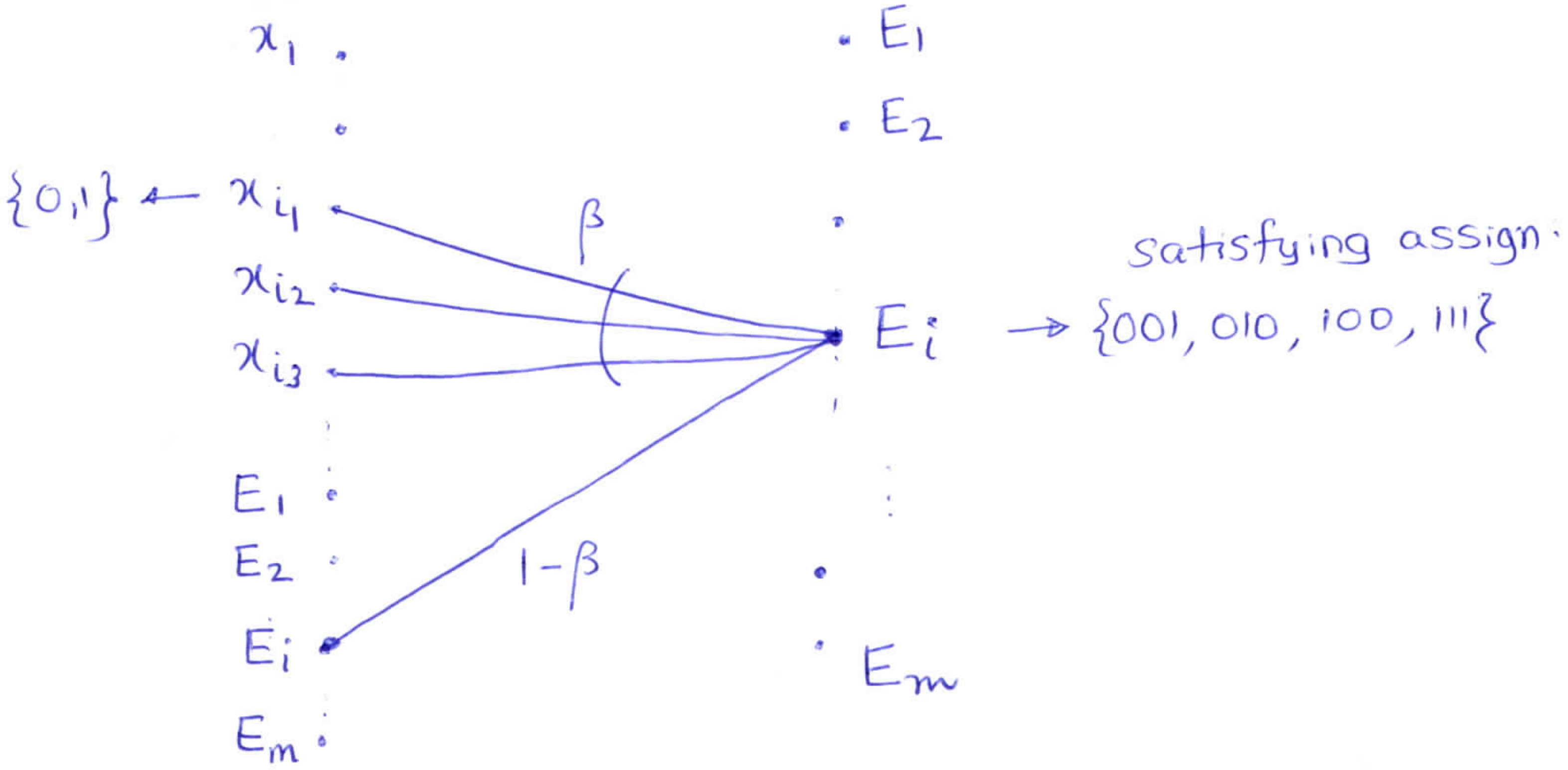
$\vdots$

E_m

[Håstad] NP-hard to distinguish whether

$\text{OPT} \geq 1 - \varepsilon$ or $\text{OPT} \leq \frac{1}{2} + \varepsilon \leq 0.6$

Smooth Game (Label cover)



- Pick E_i at random from R.H.S.
- From L.H.S.
  - Pick E_i w.p. $1-\beta$
  - Pick x_{i_1}, x_{i_2} or x_{i_3} w.p. $\frac{\beta}{3}$ each.

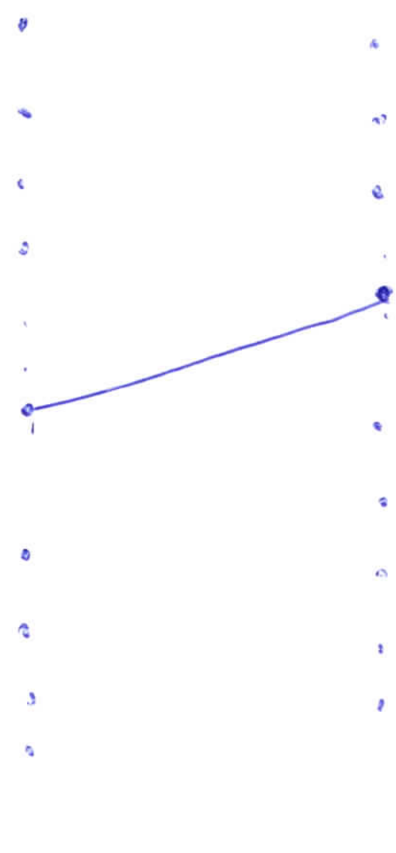
<u>Fact</u> NP-hard to distinguish whether

$\text{OPT} \geq 1-\varepsilon$ <u>or</u> $\text{OPT} \leq 1-\Omega(\beta)$.

— x —

<u>Smooth Parallel Repetition t-wise</u>

"mixed tuples"



$(E_1, E_2, \ldots, E_t)$

$(E_1, x_i, \ldots, E_t)$

$\approx (1-\beta)t$ eqns

βt vars

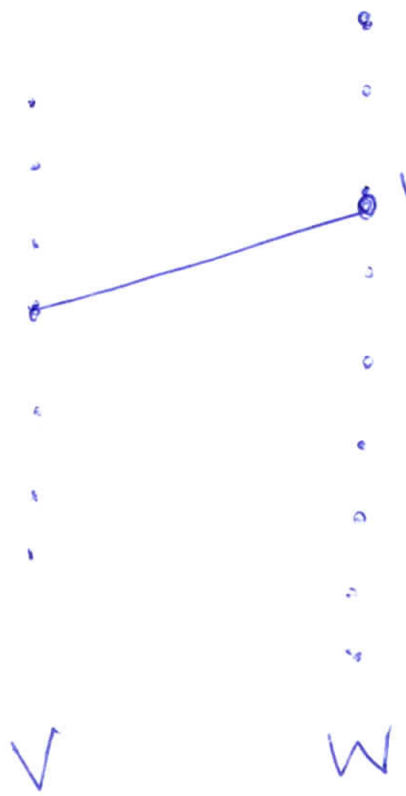
Abstractly

$w \to \sigma(w) \in \mathbb{F}_2^{k}$

$k = 3t$

$\mathbb{F}_2^{k'} \ni \sigma(v) \leftarrow v$

$k' \approx 3(1-\beta)t + \beta t.$

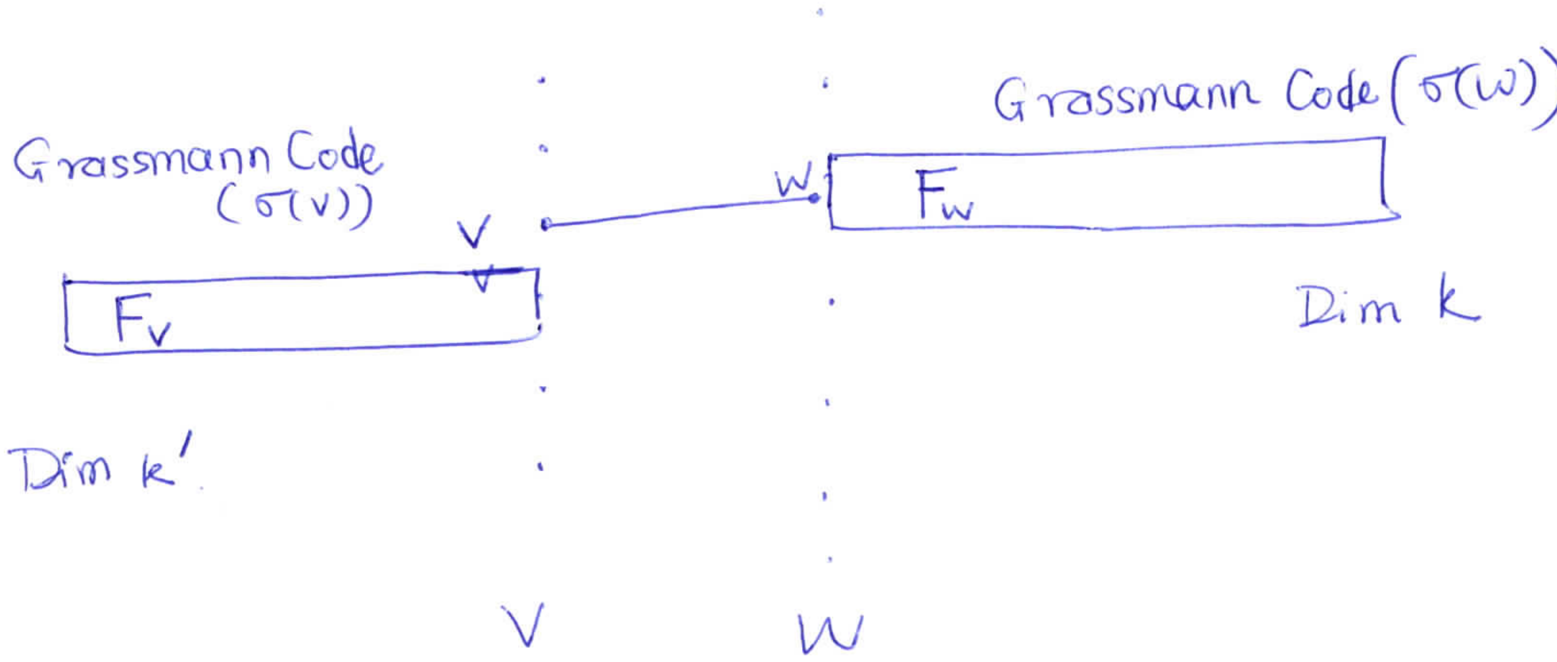


- Check that $\sigma(w)$ satisfies linear eqns.
- $\sigma(v)$ is <u>substring</u> of $\sigma(w)$.
- Think of $\beta = \frac{1}{\sqrt{t}}$ $\quad \boxed{k' \approx k}$ close
- NP-hard to distinguish whether

$$\text{OPT} \geq 1 - \varepsilon t \geq 1 - o_{\varepsilon}(1)$$

<u>or</u> $$\text{OPT} \leq (1 - \Omega(\beta))^{t} \leq o_{t}(1).$$

"Plug-in" the Grassmann Code



- $F_w[L] = f_{\sigma(w)}|_L \quad \forall L \subseteq \mathbb{F}_2^k, \dim(L) = \ell.$

  $F_v[L'] = f_{\sigma(v)}|_{L'} \quad \forall L' \subseteq \mathbb{F}_2^{k'}, \dim(L') = \ell.$

  $\mathbb{F}_2^{k'} \subseteq \mathbb{F}_2^k$ (substrings)

<u>Test</u> - Pick (L', L) s.t. $\dim(L' \cap L) = \ell - 1$.

$L' \subseteq \mathbb{F}_2^{k'}$, $L \subseteq \mathbb{F}_2^{k}$

- Accept iff $F_v[L']|_{L \cap L'} = F_w[L]|_{L \cap L'}$. ▨