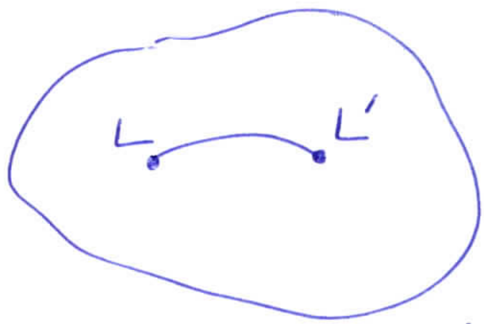
Proof of 2-to-2 Games Theorem

$$
\begin{array}{ccc}
\text{3Lin} & \xrightarrow{\hspace{4em}} & \text{2-to-2 Game} \\
& \uparrow & \\
& \text{(G) Linearity Testing Theorem} & \\
& \uparrow \checkmark & \\
& \text{(G) Expansion Theorem} &
\end{array}
$$

Grassmann Expansion Theorem

$G(V,E)$



- $L \subseteq \mathbb{F}_2^k$
- $\dim(L) = \ell$
- $\dim(L \cap L') = \ell - 1$

$\forall \delta \; \exists r, \theta$ s.t.

$\Phi(S) \leq 1 - \delta \quad \Rightarrow$

$\exists \; A, W \qquad \dim(A) = a$

$\dim(W) = k - b$

$a + b \leq r$

$$\frac{|S \cap S_{A,W}|}{|S_{A,W}|} \geq \theta \qquad S_{A,W} = \{ L \mid A \subseteq L \subseteq W \}.$$

"Toy", "Moral" version of proof

** Details imprecise **

Recall.

Eigenvalues $1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^\ell}$

Eigenspaces $J_0 \quad J_1 \quad J_2 \quad \cdots \quad J_\ell$

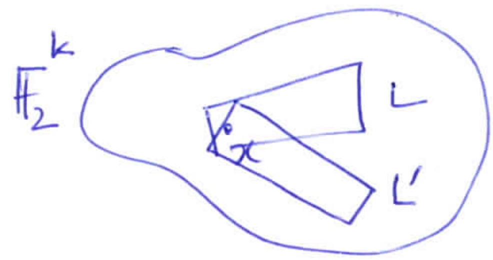
Fact Every $F : V \to \mathbb{R}$ can be written as

$$F = F_{=0} + F_{=1} + F_{=2} + \cdots + F_{=\ell}$$

$$\begin{array}{cccc} \cap & \cap & \cap & \cap \\ J_0 & J_1 & J_2 \cdots & J_\ell \end{array}$$

$$\|F\|_2^2 = \|F_{=0}\|_2^2 + \|F_{=1}\|_2^2 + \cdots + \|F_{=\ell}\|_2^2$$

Fact For $x \in \mathbb{F}_2^k$, $x \neq 0$,



$S_x = \{ L \mid x \in L \}$.

$$J_1 \approx \mathrm{Span}(\{ \mathbb{1}_{S_x} \})$$

Fact: Every $F \in J_1$ can be written as

$$F = \sum_{x \in \mathbb{F}_2^k} f(x) \cdot \mathbb{1}_{S_x}$$

- coefficients
- $f : \mathbb{F}_2^k \to \mathbb{R}$.

$$\therefore \quad F[L] = \sum_{x \in \mathbb{F}_2^k} f(x) \, \mathbb{1}_{S_x}(L)$$

$$\therefore \quad \boxed{F[L] = \sum_{x \in L} f(x).}$$

Fact $\quad \|F\|_2^2 = 2^\ell \cdot \|f\|_2^2$.

"Proof": $\quad \|F\|_2^2 = \mathbb{E}_L [F[L]^2]$

$$= \mathbb{E}_L \left[\left(\sum_{x \in L} f(x) \right)^2 \right]$$

$$= \mathbb{E}_L \left[\sum_{x \in L} f(x)^2 \right]$$

$$= 2^\ell \cdot \mathbb{E}_x [f(x)^2] = 2^\ell \|f\|_2^2 \quad \blacksquare$$

$\sum_x f(x) = 0$

(?)

Illustration

$$F = \mathbb{1}_{S_{x_0}}$$

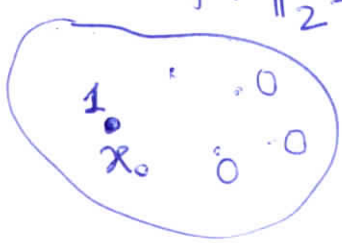
$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

$$F[L] = \sum_{x \in L} f(x)$$

$$= \begin{cases} 1 & \text{if } x_0 \in L \\ 0 & \text{otherwise} \end{cases}$$

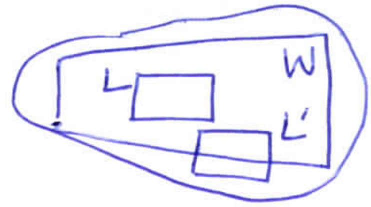
$$= \mathbb{1}_{S_{x_0}}(L).$$

$f : \mathbb{F}_2^k \to \mathbb{R}$



$$\|f\|_\infty \geq 1.$$

$\mathbb{F}_2^k$

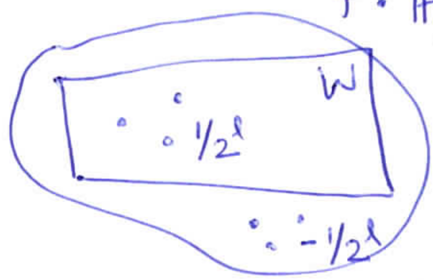


$$F = \mathbb{1}_{S_W}$$

$$\approx \frac{1}{2^l}\left(\sum_{x \in W} \mathbb{1}_{S_x} - \sum_{x \notin W} \mathbb{1}_{S_x}\right)$$

$$\therefore\ f(x) = \begin{cases} 1/2^l & \text{if } x \in W \\ -1/2^l & \text{if } x \notin W \end{cases}$$

$f : \mathbb{F}_2^k \to \mathbb{R}$



$$\therefore\ \hat{f}(W) = 1/2^l.$$

$\therefore$ f has "high" Fourier coefficient.

$$\|\hat{f}\|_\infty \geq \frac{1}{2^l}.$$

Back to Grassmann Expansion Theorem

- $G(V,E)$.
- Let $S \subseteq V$ be s.t. $\Phi(S) \leq 1-\alpha = \frac{1}{2} + o(1)$.

  $|S| = \delta |V|$, δ very small.
- We'll show that S "resembles" either some S_{x_0} or some S_W. (canonical sets w/ expansion $\approx \frac{1}{2}$)

Let $F = \mathbb{1}_S$ be indicator of S.

$$\|F\|_2^2 = \mathop{\mathbb{E}}_{L}\left[F[L]^2\right] = \delta.$$

$$\|F\|_2^2 = \|F_{=0}\|_2^2 + \|F_{=1}\|_2^2 + \|F_{=2}\|_2^2 + \cdots + \|F_{=\ell}\|_2^2$$

$$\therefore \quad \delta = \delta^2 + \|F_{=1}\|_2^2 + \|F_{=2}\|_2^2 + \cdots + \|F_{=\ell}\|_2^2$$

$$\frac{\delta}{2} \approx \delta^2 + \underbrace{\frac{1}{2}}\|F_{=1}\|_2^2 + \underbrace{\frac{1}{4}}\|F_{=2}\|_2^2 + \cdots + \underbrace{\frac{1}{2^\ell}}\|F_{=\ell}\|_2^2$$

$\approx \delta(1-\Phi(S))$

eigenvalues

$\therefore \quad \delta \approx \sum_{i=1}^{\ell}$ ("Mass" at level i)

$\frac{\delta}{2} \approx \sum_{i=1}^{\ell} \frac{1}{2^i}$ ("Mass" at level i)

$\therefore$ Almost all "mass" is at Level 1 !

$\therefore \quad F \approx F_{=1}$.

$= \sum_{x \in L} f(x)$ for some $f : \mathbb{F}_2^k \to \mathbb{R}$

$\|f\|_2^2 \approx 2^{-\ell} \cdot \|F\|_2^2$

$= 2^{-\ell} \cdot \delta$.

We'll show that

Either $\|f\|_\infty \geq \Omega(1)$ or $\|\hat{f}\|_\infty \geq \Omega(2^{-\ell})$

$\therefore \quad F = \mathbb{1}_S$ "resembles"

Either some S_{x_0} or some S_W. Formally

$\frac{|S \cap S_{x_0}|}{|S_{x_0}|} \geq \Omega(1)$ or $\frac{|S \cap S_W|}{|S_W|} \geq \Omega(1)$ (Skipped).

"Proof".

$$\delta = \|F\|_2^2 \qquad \mathbb{E}[F[L]^2] \qquad F \text{ Boolean}$$

$$= \|F\|_4^4 \qquad \mathbb{E}[F[L]^4]$$

$$\approx \|F_{=1}\|_4^4$$

$$= \mathbb{E}_L\left[F_{=1}[L]^4\right] \qquad \dim(L) = l.$$

$$= \mathbb{E}_L\left[\left(\sum_{x \in L} f(x)\right)^4\right]$$

$$= \mathbb{E}_L\left[\sum_{x,y,z,w \in L} f(x) \cdot f(y) f(z) f(\omega)\right]$$

$$= 2^l \cdot \mathbb{E}_x\left[f(x)^4\right] \quad \text{— Type 1}$$

$$+ 2^{3l} \mathbb{E}_{x,y,z}\left[f(x) f(y) f(z) f(x+y+z)\right] \quad \text{— Type 2}$$

$$+ \ 2^{2\ell} \ \mathbb{E}_x\left[f(x)^2\right] \ \mathbb{E}_z\left[f(z)^2\right] \qquad \text{— Type 3}$$

$$\vdots$$

— Type 6.

$\therefore$ At least one of the types contributes at least $\delta/6$.

<u>Type 1</u>

$$\begin{aligned}
\frac{\delta}{6} &\leq 2^{\ell} \cdot \mathbb{E}_x\left[f(x)^4\right] \\
&\leq 2^{\ell} \cdot \mathbb{E}_x\left[f(x)^2\right] \cdot \|f\|_\infty^2 \\
&= 2^{\ell} \ \|f\|_2^2 \ \|f\|_\infty^2 \\
&= 2^{\ell} \cdot 2^{-\ell} \cdot \delta \cdot \|f\|_\infty^2
\end{aligned}$$

$$\therefore \quad \frac{1}{\sqrt{6}} \leq \|f\|_\infty \qquad !!$$

Type 2

$$\frac{\delta}{6} \leq 2^{3\ell} \cdot \mathop{\mathbb{E}}_{\substack{x,y,z \\ \in \mathbb{F}_2^k}} \left[f(x)\, f(y)\, f(z)\, f(x+y+z) \right]$$

$$= 2^{3\ell} \cdot \sum_{W} \hat{f}(W)^4$$

- $W \subseteq \mathbb{F}_2^k$
- $\dim(W) = k-1$
- abuse of notation

$$\leq 2^{3\ell} \cdot \left(\sum_{W} \hat{f}(W)^2 \right) \cdot \| \hat{f} \|_\infty^2$$

$$= 2^{3\ell} \cdot \| f \|_2^2 \, \| \hat{f} \|_\infty^2$$

$$= 2^{3\ell} \cdot \delta \cdot 2^{-\ell} \cdot \| \hat{f} \|_\infty^2$$

$$\therefore \quad \frac{1}{\sqrt{6}} \cdot 2^{-\ell} \leq \| \hat{f} \|_\infty . \quad !!$$

$\blacksquare$