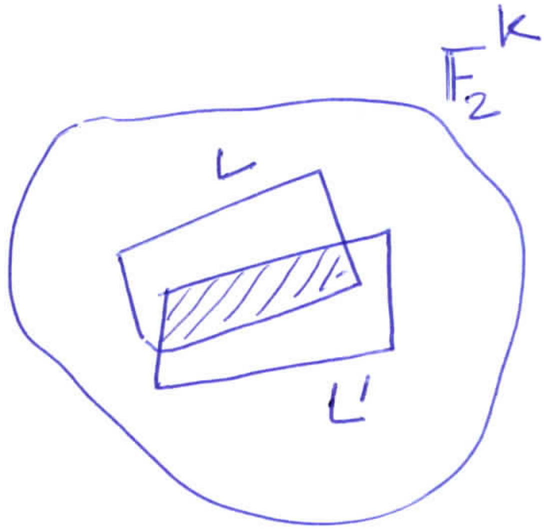
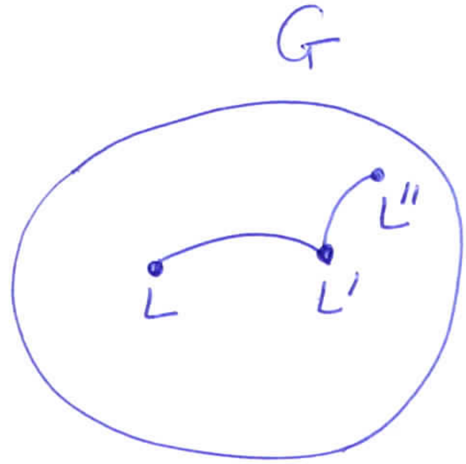
The Grassmann Graph





- $G(V, E)$
- V : all ℓ-dim subspaces L of $\mathbb{F}_2^k$.
- E : (L, L'), $\dim(L \cap L') = \ell - 1$.

Eigenvalues $\quad 1, \approx \frac{1}{2}, \approx \frac{1}{4}, \cdots, \approx \frac{1}{2^\ell}$

Multiplicities $\quad 1, \approx \begin{bmatrix} k \\ 1 \end{bmatrix}, \approx \begin{bmatrix} k \\ 2 \end{bmatrix}, \cdots, \approx \begin{bmatrix} k \\ \ell \end{bmatrix}$

Eigenspaces $\quad J_0 \quad J_1 \quad J_2 \quad \cdots \quad J_\ell$

(To be precise, $\dim(J_r) = \begin{bmatrix} k \\ r \end{bmatrix} - \begin{bmatrix} k \\ r-1 \end{bmatrix}$.)

- $\forall f \in J_r, \quad T_G f \approx \frac{1}{2^r} f.$

- Orthogonality:

  $\forall f \in J_r, f' \in J_{r'}, r \neq r', \quad \langle f, f' \rangle = 0.$

- Parseval / Fourier.

  Every $f : V \to \mathbb{R}$ can be written as

$$f = f_0 + f_1 + f_2 + \cdots + f_\ell$$

$$\begin{array}{cccc} \cap & \cap & \cdots & \cap \\ J_0 & J_1 & & J_\ell \end{array}$$

$$\|f\|_2^2 = \|f_0\|_2^2 + \|f_1\|_2^2 + \cdots + \|f_\ell\|_2^2.$$

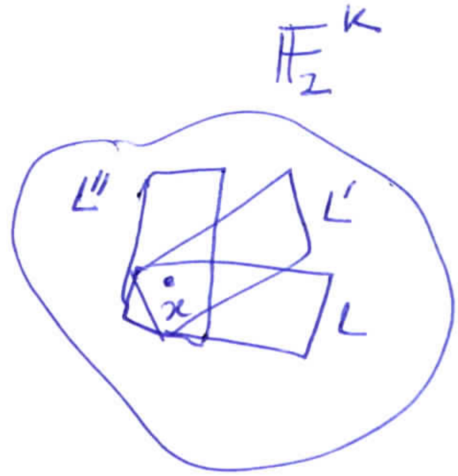
<u>Goal</u> To characterize sets $S \subseteq V$ s.t.

- $|S| = o(|V|)$ and

- $\Phi(S) \leq 0.99.$

(Note such set S is "non-random", "exceptional").

Sets with $\Phi(S) \approx \frac{1}{2}$



For $x \in \mathbb{F}_2^k$, $x \neq 0$, let

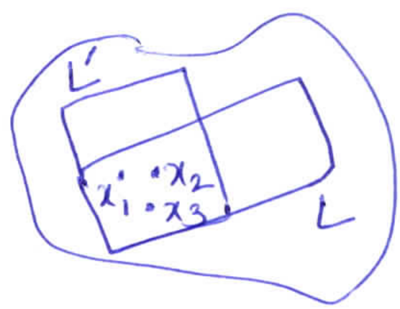
$$S_x = \{ L \mid x \in L \}.$$

Claim $\Phi(S_x) \approx \frac{1}{2}$.

Proof For any $L \in S_x$, $x \in L$,

$$\Pr_{L' : (L, L') \in E} [L' \in S_x] = \Pr_{L'}[x \in L'] \approx \frac{1}{2}.$$

Sets with $\Phi(\ ) \approx 1 - \frac{1}{2^a}$



For $x_1, x_2, \ldots, x_a$, lin indep

$$S_{x_1, \ldots, x_a} = \{ L \mid x_1, \ldots, x_a \in L \}$$

$$S_A = \{ L \mid A \subseteq L \}, \quad \dim(A) = a.$$

Claim $\Phi(S_{x_1, \ldots, x_a}) \approx 1 - \frac{1}{2^a}$ (same proof).

Roughly Speaking

Recall :

Eigenvalues	1,	$\approx \frac{1}{2}$,	$\approx \frac{1}{4}$,	$\cdots$,	$\approx \frac{1}{2^\ell}$
Eigenspaces	J_0	J_1	J_2	$\cdots$	J_ℓ
dimensions	1	$\approx \begin{bmatrix} k \\ 1 \end{bmatrix}$	$\approx \begin{bmatrix} k \\ 2 \end{bmatrix}$	$\cdots$	$\approx \begin{bmatrix} k \\ \ell \end{bmatrix}$

Roughly - $J_0 \equiv 1$

- J_1 : spanned by indicators of sets S_x.

  Eigenvalue $\approx$ Expansion $\approx \frac{1}{2}$.

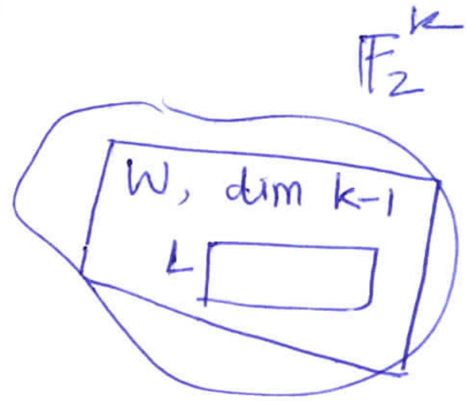
- J_r : spanned by indicators of $S_{x_1, \ldots, x_r}$

  Eigenvalue $\approx$ 1 $-$ Expansion $\approx \frac{1}{2^r}$.

(Strictly speaking

- $\mathrm{Span}(\{\mathbb{1}_{S_{x_1, \ldots, x_r}}\}) = J_0 \oplus J_1 \oplus \cdots \oplus J_r$.
- $\dim(J_r) = \begin{bmatrix} k \\ r \end{bmatrix} - \begin{bmatrix} k \\ r-1 \end{bmatrix}$. No easy description of eigenvectors.)

Dual sets with $\Phi(\ ) \approx \frac{1}{2}$



For $y \in \mathbb{F}_2^k$, $y \neq 0$

$S_{y^\perp} = \{ L \mid L \perp y \}$.

$S_W = \{ L \mid L \subseteq W \}$ $\quad W = y^\perp$, dim $k-1$.

Claim $\quad \Phi(S_{y^\perp}) \approx \frac{1}{2}$.

Proof $\quad$ Fix $L \subseteq W = y^\perp$, $\quad L \in S_{y^\perp}$.

$$\Pr_{L' : (L, L') \in E} [L' \in S_{y^\perp}] = \Pr_{L'} [L' \subseteq W] \approx \frac{1}{2}.$$

Dual sets with $\Phi(\ ) \approx 1 - \frac{1}{2^b}$

For $y_1, \ldots, y_b \in \mathbb{F}_2^k$, lin indep

$S_{y_1^\perp, \ldots, y_b^\perp} = \{ L \mid L \perp \{ y_1, \ldots, y_b \} \}$

$S_W = \{ L \mid L \subseteq W \}$ $\quad W = \{ y_1, \ldots, y_b \}^\perp$.

$\Phi(S_{y_1^\perp, \ldots, y_b^\perp}) \approx 1 - \frac{1}{2^b}$ $\quad$ (same proof).

In general:

- $\underbrace{x_1, x_2, x_a}, \underbrace{y_1, \dots, y_b}$ lin indep.

- $S_{x_1,\dots,x_a, y_1^{\perp},\dots,y_b^{\perp}} = \left\{ L \;\middle|\; \begin{array}{l} x_1,\dots,x_a \in L \\ L \perp \{y_1,\dots,y_b\} \end{array} \right\}.$

$$\Phi(\quad) \approx 1 - \frac{1}{2^{a+b}}$$

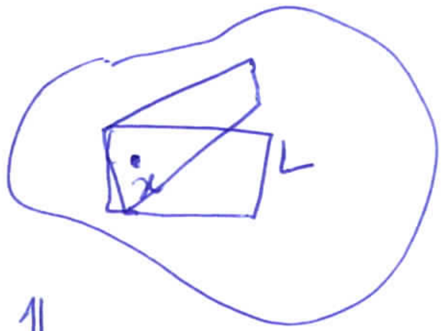
- Alternate notation
  - $A \subseteq W$
  - $\dim(A) = a, \quad \dim(W) = k-b.$
  - $S_{A,W} = \{ L \mid A \subseteq L \subseteq W \}.$

$$\Phi(S_{A,W}) \approx 1 - \frac{1}{2^{a+b}}.$$

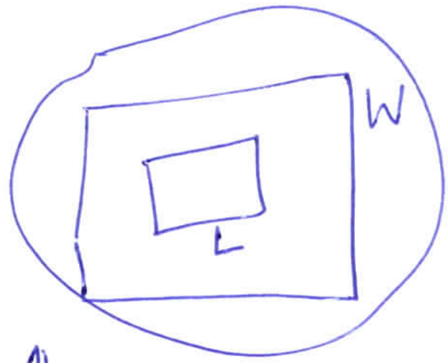
(Roughly) Grassmann Expansion Theorem

Sets $S_{A,W}$, $a+b \leq r$, "essentially" characterize all sets S, $\Phi(S) \leq \underbrace{1 - \frac{1}{2^r}}_{\text{(say) } 0.99}.$

Interesting aside (Dual view of eigenspaces J_n).



$\mathbb{1}_{S_x}$



$\mathbb{1}_{S_W}$

$\dim(W) = k-1$.

- $\text{Span}(\{\mathbb{1}_{S_x}\}) =$ Eigenspace w/ eigenvalue $\approx \frac{1}{2}$.

  $\mathbb{1}_{S_W}$ also has expansion/eigenvalue $\approx \frac{1}{2}$.

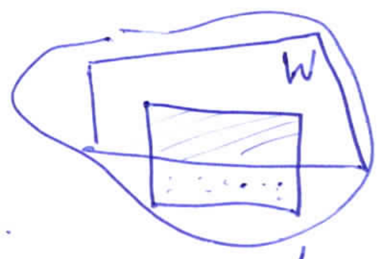
- Does $\mathbb{1}_{S_W} \in \text{span}(\{\mathbb{1}_{S_x}\})$?

- <u>YES!</u>

$$\mathbb{1}_{S_W} \approx \frac{1}{2^\ell}\left(\sum_{x \in W} \mathbb{1}_{S_x} - \sum_{x' \notin W} \mathbb{1}_{S_{x'}}\right).$$

<u>Proof</u> $= 1$, $L \subseteq W$. $\frac{1}{2^\ell} \cdot 2^\ell = 1$.

$= 0$, $L \not\subseteq W$. $\frac{1}{2^\ell}(2^{\ell-1} - 2^{\ell-1}) = 0$.



$\blacksquare$

Side Observation

- $S \subseteq V$, $|S| = \theta |V|$,

  $\theta \approx 0.10$

$|S| = \theta |V|$.

- Then $\Phi(S) \leq 1 - \theta$.

Proof For every $\ell - 1$ dimension subsp T, let $\theta_T = \Pr_{T \subseteq L} [L \in S]$. $\dim(L) = \ell$.

Then $\mathbb{E}_T [\theta_T] = \Pr_L [L \in S] = \theta$.

On the other hand,

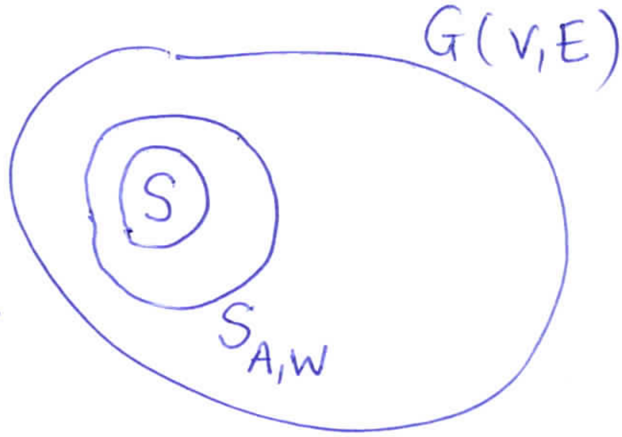
$$\Pr_{(L,L') \in E} [L \in S, L' \in S] = \Pr_{\substack{T \\ T \subseteq L, T \subseteq L'}} [L \in S, L' \in S]$$

$$= \mathbb{E}_T [\theta_T^2]$$

$$\geq (\mathbb{E}_T [\theta_T])^2 \geq \theta^2.$$

$$\therefore 1 - \Phi(S) = \frac{\Pr_{(L,L')} [L \in S, L' \in S]}{\Pr_L [L \in S]} \geq \frac{\theta^2}{\theta} = \theta. \quad \blacksquare$$

Corollary



- $S_{A,W}$, $\Phi(\ ) = 1 - \frac{1}{2^{a+b}}$.
- $S \subseteq S_{A,W}$, $\frac{|S|}{|S_{A,W}|} = \theta$. $\theta \approx 0.10$.

Then $\Phi(S) \leq 1 - \theta \cdot \frac{1}{2^{a+b}}$.

Proof Consider random $L \in S$, $(L, L') \in E$.

$$\Pr[L' \in S_{A,W}] \approx \frac{1}{2^{a+b}}.$$

$$\Pr[L' \in S \mid L' \in S_{A,W}] \geq \theta.$$ as seen.

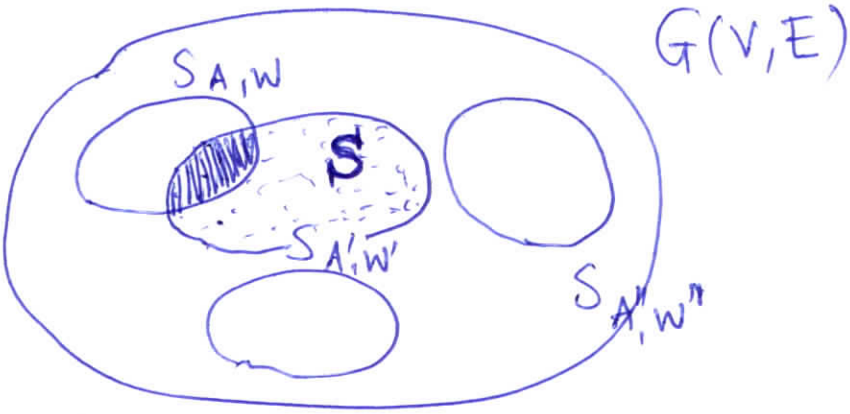
Here we use that $G^{k,l}|_{S_{A,W}} \equiv G^{k-b-a,\, l-a}$.

—— x ——

So to our "list" of sets S, $\Phi(S) \leq 0.9999$, we can include

- in addition to $S_{A,W}$ $\quad a+b \leq 10$
- any $S \subseteq S_{A,W}$, $\frac{|S|}{|S_{A,W}|} \geq \theta = 0.10$ ▨.

Grassmann Expansion Theorem



$\forall \delta \quad \exists r, \theta \quad$ s.t.

For any set $S \subseteq V$, $\Phi(S) \leq 1-\delta$,

$$\exists \quad A \subseteq W, \qquad \begin{aligned} \dim(A) &= a \\ \dim(W) &= k-b \\ a+b &\leq r \end{aligned}$$

and $\quad \dfrac{|S \cap S_{A,W}|}{|S_{A,W}|} \geq \theta.$

$\blacksquare$

(Easily leads to "structure theorem"; skipped).

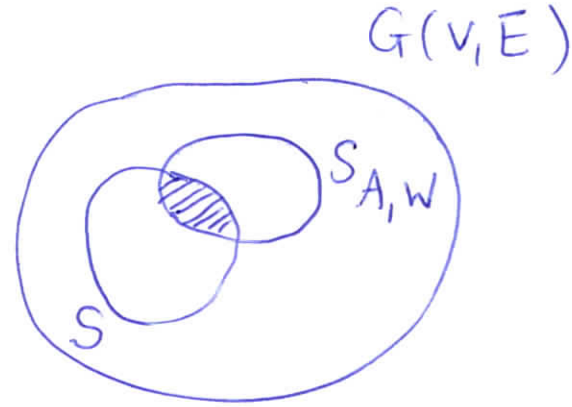
(G) Expansion Thm $\Rightarrow$ (G) Linearity Testing Thm

$\forall \delta \ \exists\ r, \theta$ s.t.

$\Phi(S) \le 1-\delta \Rightarrow$

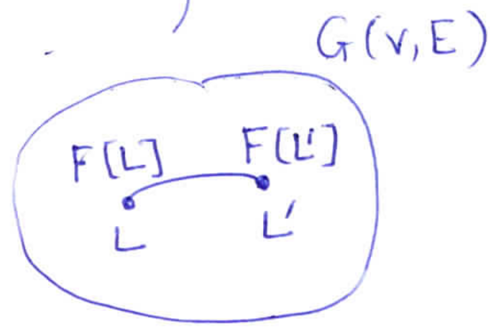
$\exists\ S_{A,W}$

$$\frac{|S \cap S_{A,W}|}{|S_{A,W}|} \geqslant \theta.$$



$\dim(A) + \operatorname{codim}(W) \le r$

$$\Downarrow$$



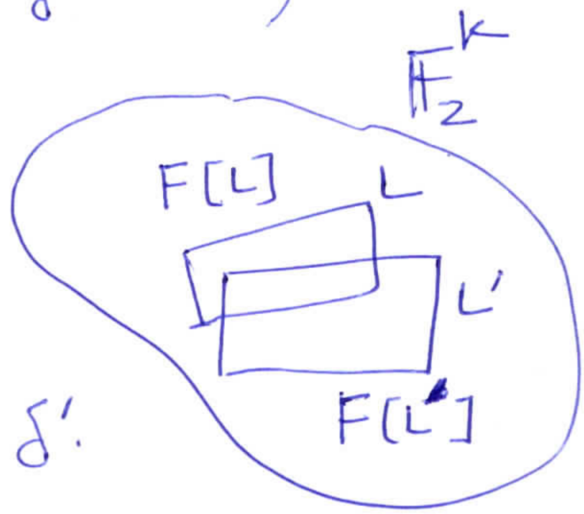
$\forall \delta \ \exists\ r, \delta'$ s.t.

$$\Pr_{(L,L')}\left[F[L], F[L'] \text{ consistent}\right] \geqslant \delta$$

$\Rightarrow \ \exists\ A, W, \quad g : \mathbb{F}_2^k \to \mathbb{F}_2$ linear

s.t.

$$\underbrace{\Pr_{L : A \subseteq L \subseteq W}}_{L \in S_{A,W}.}\left[F[L] = g|_L\right] \geqslant \delta'.$$

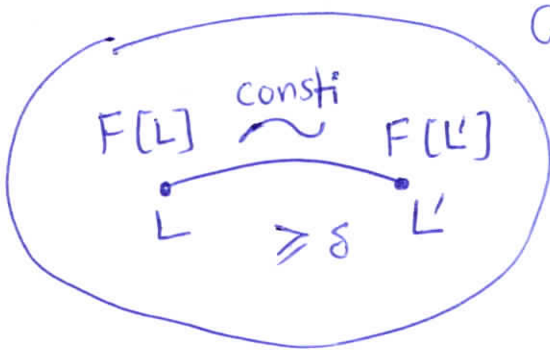


$\dim(A) + \operatorname{codim}(W) \le r.$

Proof Let g be a $\underline{\overset{!!!}{\text{random}}}$ $g : \mathbb{F}_2^k \to \mathbb{F}_2$ linear f^n.

$$S_g = \{ L \mid F[L] = g|_L \}.$$

G(V, E)



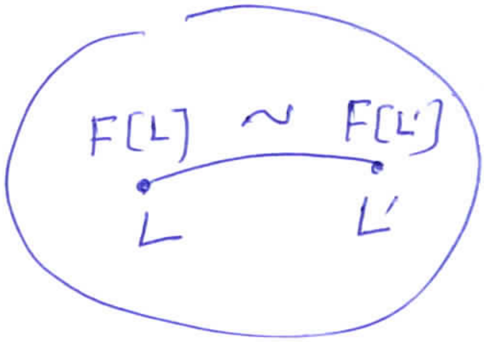
Obs 1 $\mathbb{E}[S_g] = 2^{-l}$ (relative to $|V|$). size

Obs 2 $\mathbb{E}[e(S_g)] \geq 2^{-l} \cdot \delta \cdot \frac{1}{2}$. $e(S_g)$ = fraction of edges in S_g.

Proof Let (L, L') be any edge on which $F[L]$, $F[L']$ are consistent.

Fraction of such edges is $\geq \delta$.

We consider the probability that both $L, L' \in S_g$. (over choice of g)



- $\Pr[L \in S_g] = \Pr[F[L] = g|_L] = 2^{-1}$,
- Conditional on $L \in S_g$,

$$\Pr[L' \in S_g] = \Pr[F[L'] = g|_{L'}]$$

$$= \frac{1}{2}$$

Since $F[L']|_{L \cap L'} = F[L]|_{L \cap L'} = g|_{L \cap L'}$ already.

$$\therefore \mathbb{E}[e(S_g)] \geq 2^{-1} \cdot \delta \cdot \frac{1}{2}.$$

$\blacksquare$

Since $\mathbb{E}[S_g] = 2^{-1}$,

$\exists\, g^*$ s.t. $\Phi(S_{g^*}) \leq 1 - \frac{\delta}{2}$.

By (G) Expansion theorem

$\exists\ A, W \qquad \dim(A) + \operatorname{codim}(W) \leq r$

$\underline{\text{s.t.}}$

$$\frac{|S_{g^*} \cap S_{A,W}|}{|S_{A,W}|} \geq \theta$$

$$\therefore \Pr_{L \in S_{A,W}} [L \in S_{g^*}] \geq \theta$$

$$\therefore \Pr_{L \in S_{A,W}} [F[L] = g^*|_L] \geq \theta$$

$\blacksquare$