Graphs, Eigenvalues, Expansion

- Graph $G(V, E)$ $\qquad |V| = n$.
- Regular, degree d.
- Normalized adjacency matrix:

$$T_G(u,v) = \begin{cases} 1/d & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$T_G = \begin{array}{cc} \begin{array}{cccc} & & u & \end{array} & \\ \begin{bmatrix} 0 & & & 1/d \\ & 0 & & \\ & & 1/d & \\ 0 & & & \\ 1/d & & & 0 \end{bmatrix}_{n \times n} & \begin{array}{c} \\ \\ v \\ \\ \\ \end{array} \end{array}$$

- Eigenvector $f : V \to \mathbb{R}$ w/ eigenvalue λ.

$$\lambda \begin{bmatrix} \vdots \\ f(v) \\ \vdots \end{bmatrix} = \begin{bmatrix} & & \\ & T_G & \\ & & \end{bmatrix} \begin{bmatrix} \vdots \\ f(v) \\ \vdots \end{bmatrix} \begin{array}{c} \\ v \\ \\ \end{array}$$

- $T_G f = \lambda f$.

- Eigenvalues (assume non-negative)

$$\underbrace{\lambda_1 = 1}_{} \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n \geq 0.$$

Corresponds to function $f_1 \equiv \mathbb{1}$.

With corresponding eigenvectors

$$f_1 \quad f_2 \quad f_3 \quad \cdots \quad f_n.$$

which are an orthonormal set/basis.

- $$\langle f, g \rangle = \mathop{\mathbb{E}}_{v \in V} [f(v)\, g(v)].$$

$$\|f\|_2^2 = \mathop{\mathbb{E}}_{v \in V} [f(v)^2].$$

- $$\langle f_i, f_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Any $f: V \to \mathbb{R}$ can be written as

$$f = \sum_{i=1}^{n} \alpha_i f_i.$$

We will usually be interested in functions f that is characteristic function of subset $S \subseteq V$.

$$f(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\|f\|_2^2 = \delta$ if $|S| = \delta |V|$:

<u>Note</u>

$$T_G f(v) = \sum_{w \in V} T_G(w,v) \cdot f(w)$$

$$= \frac{1}{d} \sum_{w:(w,v) \in E} f(w)$$

$$= \mathbb{E}_{w \sim v} [f(w)].$$

<u>Def</u> Expansion of set $S \subseteq V$.

$$\Phi(S) = \Pr_{\substack{v \in S \\ w \sim v}} [w \notin S].$$

Eigenvalues & Expansion

We will be interested in sets $S \subseteq V$, $|S| = \delta |V|$ and δ small.

Let $f = \mathbb{1}_S$ be characteristic f^n of S.

Write $f = \sum_{i=1}^{n} \alpha_i f_i$

f_1	f_2	$\cdots$	f_n
$\lambda_1 = 1$	λ_2	$\cdots$	λ_n .

$$1 - \Phi(S) = \Pr_{\substack{v \in V \\ w \sim v}} [w \in S]$$

$$= \frac{\Pr_{v,\, w \sim v} [v \in S,\, w \in S]}{\Pr_{v} [v \in S]}$$

$$= \frac{1}{\delta} \cdot \mathbb{E}_{v,\, w \sim v} \left[f(v) f(w) \right]$$

$$\therefore \delta \cdot (1 - \Phi(S)) = \mathbb{E}_{v} \left[f(v) \cdot \mathbb{E}_{w \sim v} [f(w)] \right].$$

$$\therefore\ \delta\,(1-\Phi(S)) = \mathop{\mathbb{E}}_{v}\left[f(v)\cdot T_G f(v)\right]$$

$$= \langle f, T_G f\rangle$$

$$= \left\langle \sum_{i=1}^{n} \alpha_i f_i\ ,\ T_G \cdot \sum_{i=1}^{n} \alpha_i f_i \right\rangle$$

$$= \left\langle \sum_{i=1}^{n} \alpha_i f_i\ ,\ \sum_{i=1}^{n} \alpha_i \lambda_i f_i \right\rangle$$

$$= \sum_{i=1}^{n} \alpha_i^2 \lambda_i$$

$$\underbrace{\lambda_1\ \lambda_2 \cdots}_{>\varepsilon}\ \underbrace{\lambda_k \cdots \lambda_n}_{\leqslant \varepsilon}$$

$$\leqslant \sum_{i=1}^{k-1} \alpha_i^2 + \varepsilon \cdot \sum_{i \geqslant k} \alpha_i^2$$

$$\sum_{i=1}^{n} \alpha_i^2 = \|f\|_2^2 = \delta.$$

$$= \gamma\,\delta + \varepsilon\,(1-\gamma)\,\delta.$$

Let $\gamma\,\delta = \sum_{i=1}^{k-1} \alpha_i^2$.

$$\therefore\ 1-\Phi(S) \leqslant \gamma + \varepsilon.$$

$\therefore\ \Phi(S) \to 1 \equiv$ Almost all "Fourier mass" is on "high levels."