Proof of 2-to-2 Games Theorem

Recall Linearity Test

- Given $f : \mathbb{F}_2^k \to \mathbb{F}_2$
- Test $f(x) \oplus f(y) = f(x \oplus y)$
- If $\Pr[\text{Accept}] \geq \frac{1}{2} + \varepsilon$ non-trivial

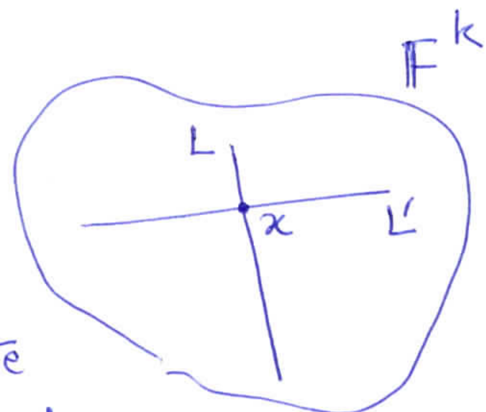
  then $\exists\, g : \mathbb{F}_2^k \to \mathbb{F}_2$, linear

  s.t. $\Pr_x[f(x) = g(x)] \geq \frac{1}{2} + \varepsilon'$ non-trivial agreement w/ codeword

- Bounded list decoding
  - Let $g_1, g_2, \dots, g_r$ be all that satisfy.
  - Then $r \leq O(1/\varepsilon'^2)$.

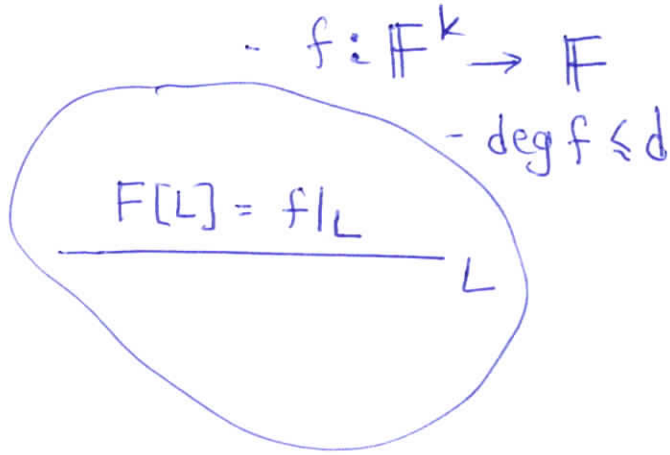
Recall Low degree Test.

- Given F : Lines $\to$ Deg d univariate polynomials



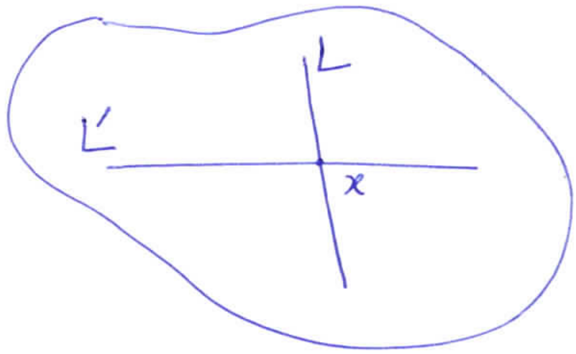
- To test that

$$F[L] = f|_L \quad \forall L.$$



- Test. Accept iff

$$F[L](x) = F[L'](x).$$



- $|\mathbb{F}| \geq \left(\frac{kd}{\delta}\right)^{O(1)}$

- If $\Pr[\text{accept}] \geq \delta$ <u>non-trivial</u>

then $\exists\, g : \mathbb{F}^k \to \mathbb{F}$, $\deg g \leq d$

s.t. $\Pr_L [F(L) = g(L)] \geq \delta'$

$\Pr_L [F[L] = g|_L] \geq \delta'$.

non-trival agreement w <u>codeword</u>

- <u>Bounded list decoding</u>

- Let $g_1, g_2, \ldots, g_r$ be all that satisfy

- then $r \leq (1/\delta')^{O(1)}$.

2-to-2 Games Theorem

Unique Games Conj.

$(\mathbb{F}_2^{\ell}, \oplus)$.

$\forall \varepsilon \ \exists \ell$ s.t. given a system

$$S \quad \begin{array}{c} \vdots \\ x_i \oplus x_j = b_{ij} \\ \vdots \end{array}$$

$x_1, \ldots, x_n$ variables over $\mathbb{F}_2^{\ell}$.

$b_{ij} \in \mathbb{F}_2^{\ell}$.

it is NP-hard to tell

if $OPT(S) \geq 1-\varepsilon$ or $OPT(S) \leq \varepsilon$.

2-to-2 Games Theorem

$\forall \varepsilon \ \exists \ell$ s.t. given a constraint system

$$\tilde{S} \quad \begin{array}{c} \vdots \\ x_i \oplus x_j \in \{b_{ij}, b'_{ij}\} \\ \vdots \end{array}$$

$b_{ij}, b'_{ij} \in \mathbb{F}_2^{\ell}$

it is NP-hard to tell

if $OPT(\tilde{S}) \geq 1-\varepsilon$ or $OPT(\tilde{S}) \leq \varepsilon$.

Recall :) 2-to-2 Games Theorem implies.

- Unique Game$_{\frac{1}{2}, \varepsilon}$ is NP-hard.

- Max-Cut$_{\frac{1}{2}+\varepsilon,\ \frac{1}{2}+\frac{\varepsilon}{\log(1/\varepsilon)}}$ is NP-hard.

- It is NP-hard to color (almost) 4-colorable graph with million colors.

- Intermediate Complexity CSP.

Unique Game$_{\frac{1}{2}, \varepsilon}$ can be solved in time $2^{n^{\varepsilon'}}$ but not in time $2^{n^{\varepsilon''}}$.

Proof of 2-to-2 Games Theorem

[Hästad]
3-LIN $\longrightarrow$ 2-to-2 Games

$\uparrow$

(Grassmann) Linearity Testing Theorem

$\uparrow$

(Grassmann) Expansion Theorem

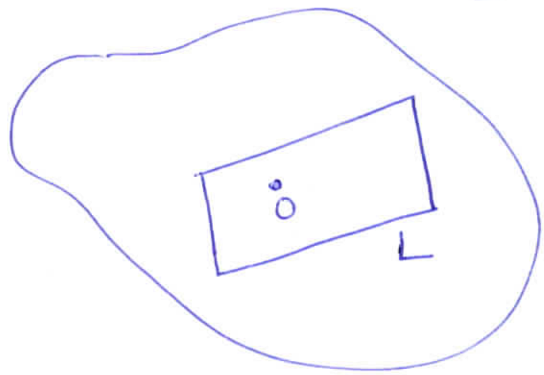
G.L.T. Theorem

- "Test" if $f : \mathbb{F}_2^k \to \mathbb{F}_2$ is linear.
- 2 queries, 2-to-2 predicate.
- NO bounded list decoding.

  NO 'standard' decoding.

Grassmann Linearity Test

$f : \mathbb{F}_2^k \to \mathbb{F}_2$



Note If f is linear,

$L \subseteq \mathbb{F}_2^k$ is a subspace,

$\dim(L) = \ell$, then

$f|_L$ is linear (and can be specified by an element from 2^ℓ-sized alphabet).

Grassmann Encoding

- To encode $f : \mathbb{F}_2^k \to \mathbb{F}_2$, linear,
- Write down $\forall\, L \subseteq \mathbb{F}_2^k$, $\dim(L) = \ell$,

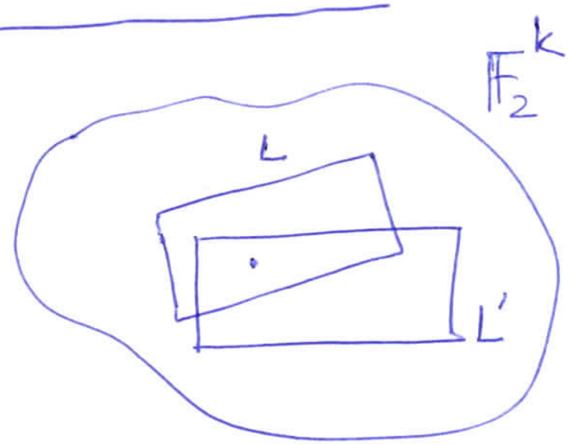
$$F[L] = f|_L .$$

Aside. Length of code $=$ # ℓ-dim subspaces of $\mathbb{F}_2^k$

$$= \frac{(2^k-1)(2^k-2)(2^k-4)\cdots(2^k-2^{\ell-1})}{(2^\ell-1)(2^\ell-2)(2^\ell-4)\cdots(2^\ell-2^{\ell-1})} = \begin{bmatrix} k \\ \ell \end{bmatrix}.$$

Grassmann Linearity Test

$1 \ll l \ll k$.



- Given $F[L]$ $\forall L$, $\dim(L) = l$.
- Pick pair (L, L') at random s.t.

  $\dim(L \cap L') = l-1$.
- Accept iff

  $$F[L]\big|_{L \cap L'} = F[L']\big|_{L\cap L'}.$$

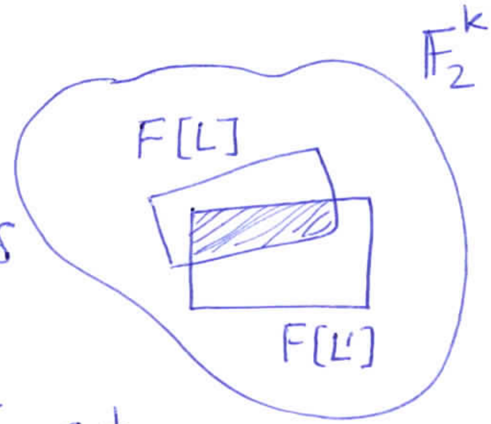
Completeness If $F[L] = f|_L$ $\forall L$ for some $f : \mathbb{F}_2^k \to \mathbb{F}_2$, linear

then $\Pr[\text{Accept}] = 1$.

2-to-2-ness Fix (L, L').

For every answer to L (i.e. $F[L]$), there are exactly two answers to L' (i.e. $F[L']$), for the test to accept. (Why?).

Soundness Speculation



If $\Pr_{(L,L')}\left[F[L],\ F[L'] \text{ consistent}\right] \geq \delta$

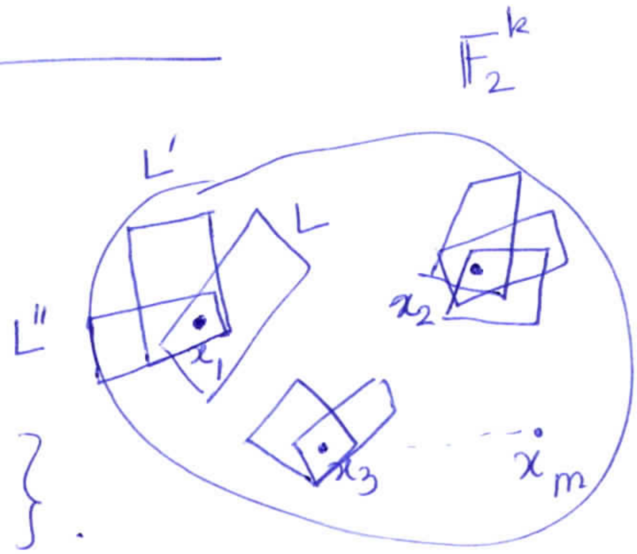
then $\exists$ global, linear $g: \mathbb{F}_2^k \to \mathbb{F}_2$ s.t.

$$\Pr_{L}\left[F[L] = g|_L \right] \geq \delta'.$$

NO!

———— x ————

Counter-example



$m \approx \frac{2^k}{2^l}$

- For $i = 1, 2, \ldots, m$,

  $S_i = \left\{ L \;\middle|\; \begin{array}{l} \dim(L) = l \\ x_i \in L \end{array} \right\}.$

- Assume S_i pairwise disjoint (for simplicity).

- $S_1 \cup S_2 \cdots \cup S_m$ cover $\Omega(1)$ fraction of l-dim subspaces.

-

- $f_1, f_2, \ldots, f_m : \mathbb{F}_2^k \to \mathbb{F}_2$ be distinct, linear.

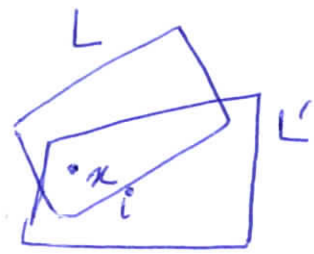
- Let
$$F[L] = \begin{cases} f_i|_L & \text{if } L \in S_i \\ \text{Random} & \text{othw.} \end{cases}$$

- <u>Note</u> $\nexists$ global, linear $f : \mathbb{F}_2^k \to \mathbb{F}_2$ s.t.
$$\Pr_L [F[L] = f|_L] \geq \delta'.$$

<u>Note</u>. $\Pr[\text{Test passes}] \geq \Omega(1)$.

<u>Proof</u>
- Pick (L, L') at random.
- $L \in S_i$ for some i w.p. $\Omega(1)$.
- $x_i \in L$.

  Consider random L',
  $\dim(L \cap L') = l - 1$.



- $x_i \in L'$ w.p. $\frac{1}{2}$
- $\therefore$ w.p. $\Omega(1)$, L, L' both $\in S_i$
  both assigned according to f_i

$\therefore$ Accept. ∎

Source of trouble

- $S_x = \{ L \mid x \in L \}$ has "expansion" $\approx \frac{1}{2}$.

(later).

- Fix to the speculation ?

  - If $\Pr[\text{Accept}] \geq \delta$ then

  $\exists\ x \in \mathbb{F}_2^k$ and global, linear $g: \mathbb{F}_2^k \to \mathbb{F}_2$

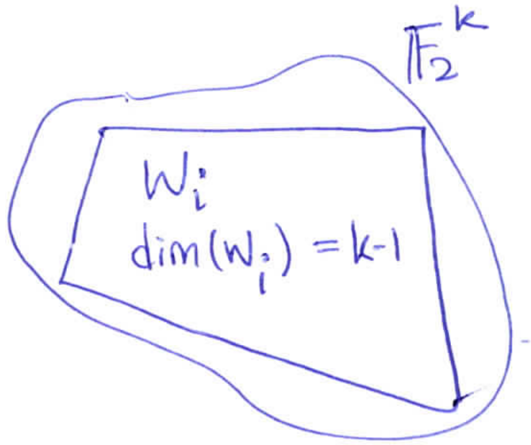
  s.t. $\Pr_{L:\, x \in L}[\, F[L] = g|_L \,] \geq \delta'$.

Zoom-in

- Fix works for the counter-example.

  Hmm ......

Dual Counter-example



- $W_1, W_2, \ldots, W_{2^\ell}$ of dim $k-1$.

- $T_j = \{ L \mid L \subseteq W_j \}$.

- Assume $T_1, \ldots, T_{2^\ell}$ disjoint

- $T_1 \cup T_2 \ldots \cup T_{2^\ell}$ is $\Omega(1)$ fraction of all L, $\dim(L) = \ell$.

- Let $f_1, \ldots, f_{2^\ell}$ be distinct, linear.

$$F[L] = \begin{cases} f_j|_L & \text{if } L \in T_j \\ \text{Random} & \text{othw.} \end{cases}$$

Note F has no significant consisteny with any global linear function.

Note $\Pr[\text{Test passes}] \geq \Omega(1)$.

- Pick (L, L') at random.
- $L \in T_j$ w.p. $\Omega(1)$.
  $L \subseteq W_j$. $L' \subseteq W_j$ w.p. $\frac{1}{2}$.
- $\therefore$ L, L' both $\subseteq W_j$ w.p. $\Omega(1)$.
  both assigned according to f_j.

$\therefore$ Accept.

$\blacksquare$

Source of trouble

- $T_W = \{ L \mid L \subseteq W \}$ has "expansion" $\approx \frac{1}{2}$ (later).

Fix to the speculation ?

- If $\Pr[\text{Accept}] \geq \delta$ then

  $\exists\ W \subseteq \mathbb{F}_2^k$ and global, linear $g : \mathbb{F}_2^k \to \mathbb{F}_2$
  $\dim(W) = k-1$

  s.t. $\Pr_{L : L \subseteq W} [\ F[L] = g|_L\ ] \geq \delta'$.

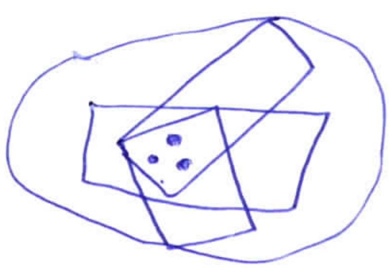
Zoom-out

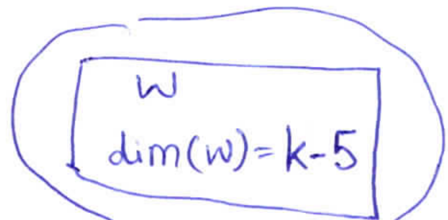
- Fix works for the dual counter-example.

Hmm .....

Similar counterexamples with

$\{x_1, x_2, x_3\}$

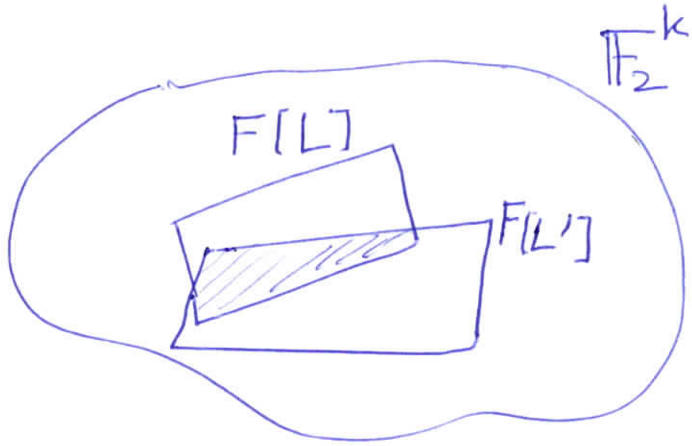




Similar fix.

Grassmann Linearity Testing Theorem

$\forall \delta \quad \exists \delta', r.$ $\qquad 1/\delta \ll \ell \ll k$



$\dim(L \cap L') = \ell - 1.$

If $\Pr_{(L,L')} [F[L], F[L'] \text{ consistent}] \geq \delta$

then $\exists A \subseteq \mathbb{F}_2^k \quad \dim(A) = \underline{a}$

$W \subseteq \mathbb{F}_2^k \quad \dim(W) = \underline{k-b}$

$a + b \leq r$

and a global, linear $g : \mathbb{F}_2^k \to \mathbb{F}_2$ s.t.

$$\Pr [F[L] = g|_L] \geq \delta'.$$

$L : A \subseteq L \subseteq W$

zoom-in + zoom-out