Hardness of Approximation

PCP Theorem to 2-to-2 Games Theorem.

<u>Def</u>: Approximation algo. for NP-hard problem P is a polytime algo. that for every instance I outputs a sol^n with value $A(I)$ s.t.

$$A(I) \geq \alpha \cdot OPT(I) \qquad (\max, \ \alpha < 1)$$

$$A(I) \leq \alpha \cdot OPT(I) \qquad (\min, \ \alpha > 1)$$

where $\alpha = \alpha(n)$ is <u>approx. ratio</u>.

<u>Def</u> Hardness of Approximation.

Under suitable complexity assumption, there is no approx. algorithm w/ ratio better than $\beta = \beta(n)$.

	Approx. ratio	Hardness factor
Bin-Packing	$1+\varepsilon$ PTAS	
3-SAT	$7/8$	$7/8+\varepsilon$
Vertex Cover	2	$2-\varepsilon$
MAX-CUT	$\alpha_{GW} = 0.878\ldots$	$\alpha_{GW}+\varepsilon$
Set Cover	$\ln n$	$(1-\varepsilon)\ln n$
Clique	n	$n^{1-\varepsilon}$

$\left.\begin{matrix} 2-\varepsilon \\ \alpha_{GW}+\varepsilon \end{matrix}\right\}$ Unique Games Conj.

NP-hard [Håstad, Feige]

<u>Plan</u>

Pre-PCP Theorem — Edge Disjoint Paths

Post-PCP Theorem — 3-LIN over $\mathbb{F}_2$

Via Unique Games Conj. — Max-Cut

Proof of 2-to-2 Games Conj. — Clique$(\frac{n}{4}, \varepsilon n)$

Proving hardness results

- Reduction from (NP-) hard problem!
- Suppose there is a (poly time) reduction

$$\text{3SAT} \longrightarrow \mathcal{P} \qquad \underline{\max}$$

$$\phi \rightsquigarrow I$$

(Yes/Completeness) $\phi \in \text{3SAT} \Rightarrow OPT(I) \geq h(n)$

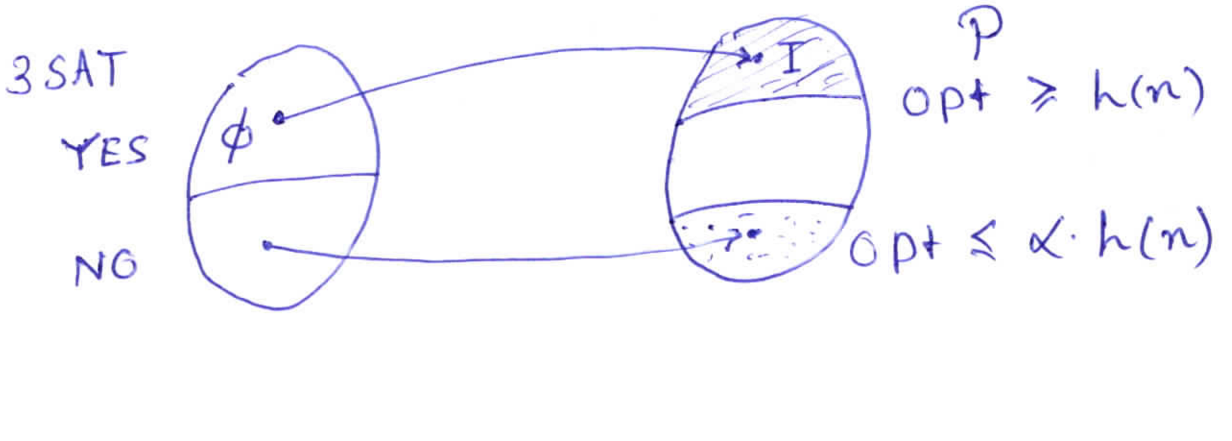
(No/ Soundness) $\phi \notin \text{3SAT} \Rightarrow OPT(I) \leq \alpha \cdot h(n)$

for some "gap" $\alpha = \alpha(n) < 1$.

Then it is NP-hard to approximate problem $\mathcal{P}$ w/ factor better than α.

Notation - $\text{Gap-}\mathcal{P}_{h(n),\ \alpha h(n)}$

- Reduction from 3SAT to $\text{Gap}\mathcal{P}_{h,\alpha h}$

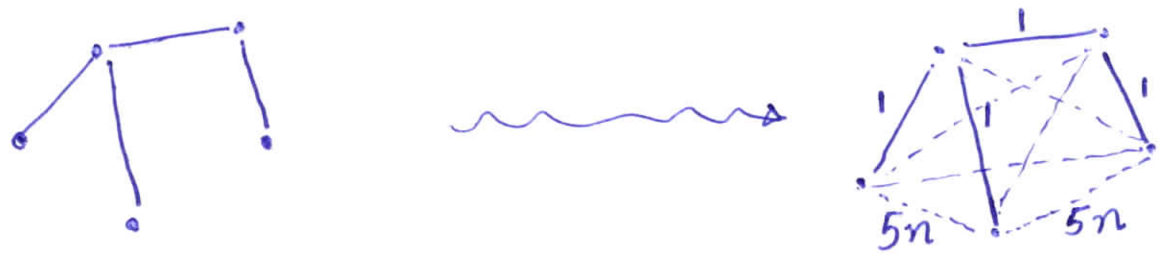


Hardness Results Pre-PCP Theorem

T. S. P. It is NP-hard to approximate TSP w/ factor 5.

Hamiltonian Cycle $\longrightarrow$ TSP

$G \rightsquigarrow K$, wt



(YES) G has Hamilt Cy. $\Rightarrow$ $\mathrm{OPT}(G) = n$

(NO) G has no Hamilt. Cy. $\Rightarrow$ $\mathrm{OPT}(G) \geq 5n$.

Done!

Hardness Results Pre-PCP Theorem

Edge Disjoint Paths

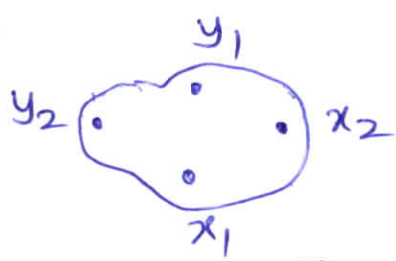
Given a directed graph $G(V,E)$ and pairs $(s_1,t_1), \ldots, (s_m,t_m)$ find max # of $s_i \rightsquigarrow t_i$ edge disjoint paths.

Theorem It is NP-hard to approximate EDP with factor $|E|^{1/2-\varepsilon}$. [GKRSY].

(Optimal: there is $|E|^{1/2}$- approx. algo. [K]).

Proof. Reduction from 2-EDP

- 2-EDP: Given dir $G(V,E)$, (x_1,y_1) (x_2,y_2)

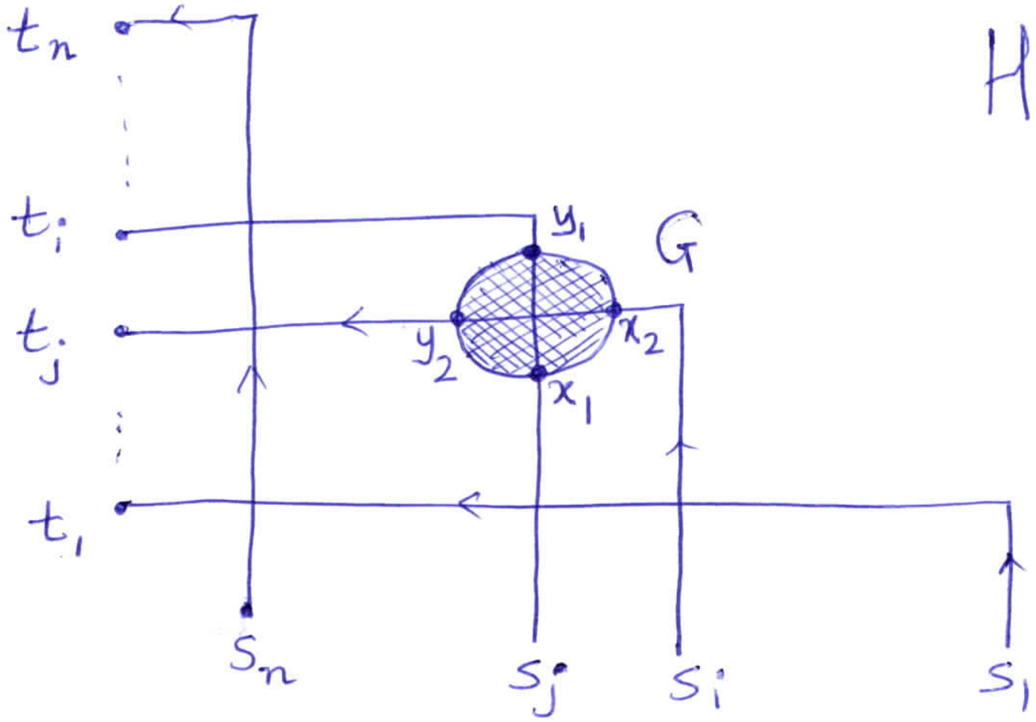


are there edge-disjoint $x_1 \rightsquigarrow y_1$ $x_2 \rightsquigarrow y_2$ paths?

NP-hard.

$$2\text{-EDP} \longrightarrow \text{EDP}$$

$$G(V,E),\ \begin{matrix}(x_1,y_1)\\(x_2,y_2)\end{matrix} \rightsquigarrow H$$



(YES) $G \in 2\text{-EDP} \Rightarrow OPT(H) = n$.

(NO) $G \notin 2\text{-EDP} \Rightarrow OPT(H) = 1$.

$\therefore$ Hardness factor $= n$.

$\#$ edges in $H = n^2 \times (\#\text{ edges in } G)$.

$\therefore$ Hardness factor $= (\#\text{ edges in } H)^{1/2 - \varepsilon}$

$\blacksquare$

What About 3SAT, Vertex Cover, Max-Cut etc ?

- Prior to PCP Theorem, it was conceivable that these have PTAS.
- PCP Theorem rules it out.
- $\therefore$ some constant threshold $\alpha_0 < 1$ (max) for $\alpha_0 > 1$ (min)

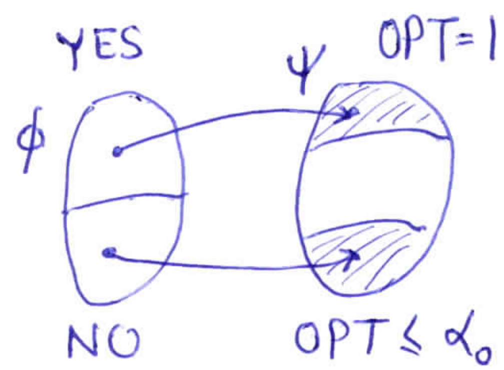
each of these problems (MAX-SNP) [PY] s.t. α_0-approx. is NP-hard.

PCP Theorem Version #1 Reduction Version:

Theorem 3SAT reduces to Gap-3SAT$_{1,\alpha_0}$.

$$3\text{SAT} \longrightarrow \text{Gap-3SAT}$$

$$\phi \rightsquigarrow \psi$$

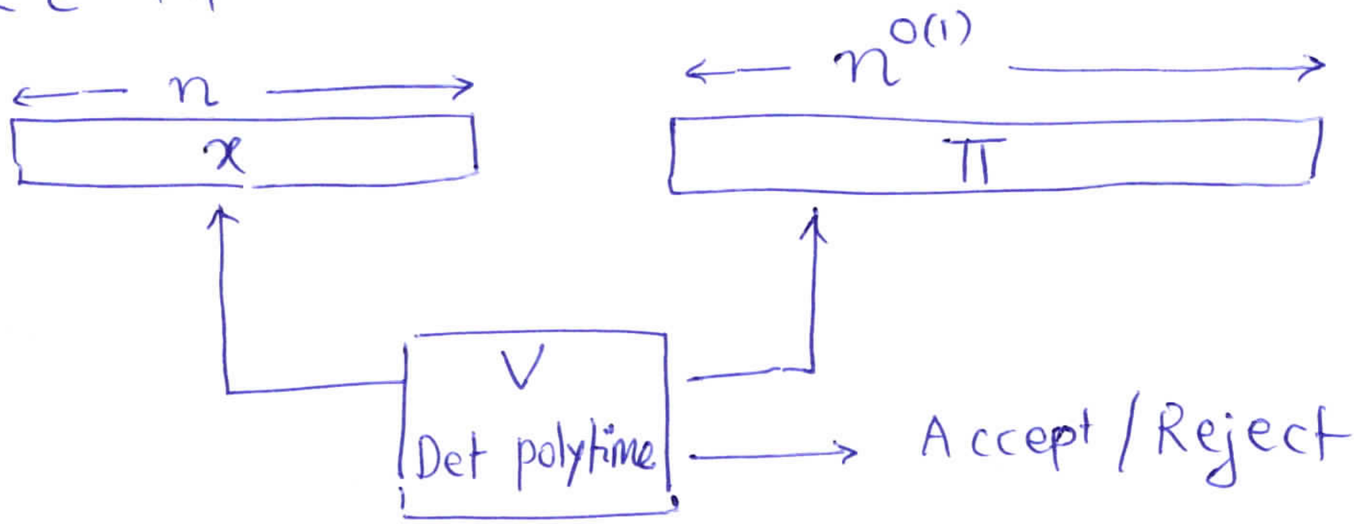


$\phi \in 3\text{SAT} \Rightarrow \text{OPT}(\psi) = 1.$

$\phi \notin 3\text{SAT} \Rightarrow \text{OPT}(\psi) \leq \alpha_0$

If ϕ is satisfiable,	ψ is satisfiable.
If ϕ is unsatisfiable,	ψ is <u>highly</u> unsatisfiable.

PCP Theorem Version #2 — Proof Checking Version

- Recall def. of NP.
- NP = Class of languages L that have membership proof that can be checked in det. polytime.
- $L \in NP$.
- 

(Comp.) $x \in L \Rightarrow \exists\, \Pi \quad V(\Pi) = \text{Accept}$.

(sound.) $x \notin L \Rightarrow \forall\, \Pi \quad V(\Pi) = \text{Reject}$.

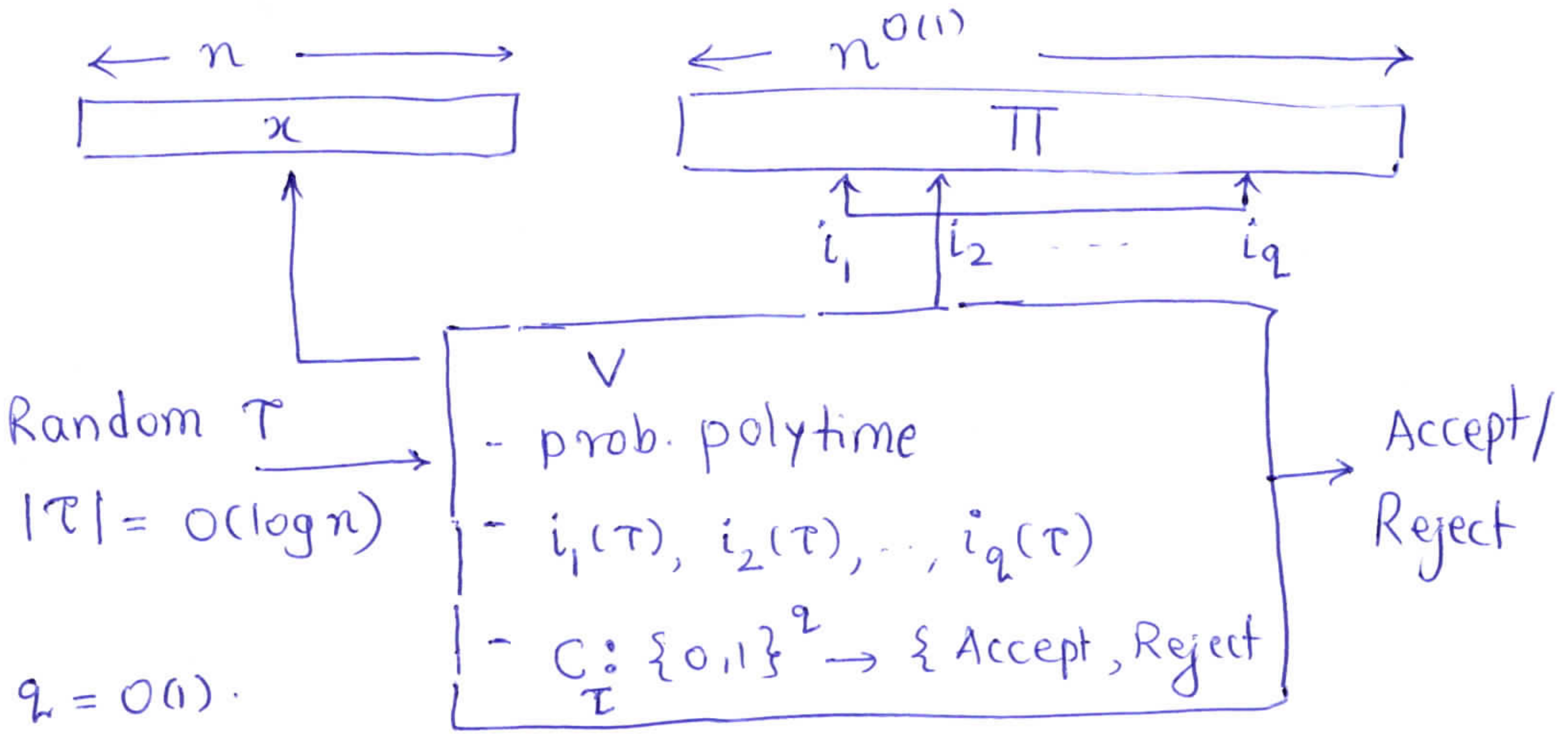
<u>E.g.</u> $L = 3SAT$ $\quad \Pi =$ (supposed) sat. assign.

$L =$ Hamiltonian cycle $\quad \Pi =$ (") Hamilt. Cycle.

PCP Theorem

Theorem Every $L \in NP$ has probabilistic polytime verifier that

- uses $O(\log n)$ random bits.
- makes $O(1)$ queries to proof (say 25).
- has "completeness" 1, "soundness" $\leq \frac{1}{2}$.



(Comp.) $x \in L \Rightarrow \exists \Pi \; \Pr_\tau [V(\Pi) = \text{Accept}] = 1.$

(Sound.) $x \notin L \Rightarrow \forall \Pi \; \Pr_\tau [V(\Pi) = \text{Accept}] \leq \frac{1}{2}$ ▨

<u>E.g.</u> A difficult theorem!

$L = 3SAT$. Instance ϕ.

- "Standard" "NP-proof" = (Supp.) assignment σ to ϕ.

- "PCP proof" Π is <u>encoding</u> of σ via an error correcting code. $|\Pi| = |\phi|^{O(1)}$.

—— x ——

We'll prove Version #1 $\Longleftrightarrow$ Version #2.

<u>Version #1</u>

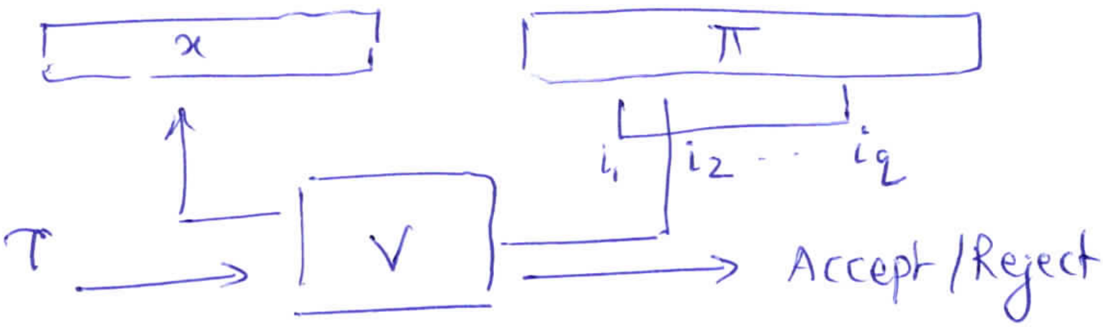
$$L \longrightarrow \text{Gap-}3SAT_{1,\alpha_0}$$

$$x \rightsquigarrow \psi$$

$$x \in L \Rightarrow OPT(\psi) = 1$$

$$x \notin L \Rightarrow OPT(\psi) \le \alpha_0 .$$

<u>Version #2</u>



$$x \in L \Rightarrow \exists\, \Pi \quad \Pr[V(\Pi) = Acc] = 1$$

$$x \notin L \Rightarrow \forall\, \Pi \quad \Pr[V(\Pi) = Acc] \le \tfrac{1}{2}.$$

Version #1 $\Rightarrow$ Version #2

- $L \in NP$. We design a verifier.
- Let $L \longrightarrow$ Gap 3SAT$_{1,\alpha_0}$ be reduction.

  $x \rightsquigarrow \psi$
- Let Π = (supp) satisfying assign. to ψ.
- V =: - Pick a random clause of ψ.

  $C = y_i \vee y_j \vee \bar{y}_k$.

  - Read y_i, y_j, y_k from Π.
  - Accept iff C is satisfied.

- Now, $\Rightarrow$ $OPT(\psi) = 1$.

  $x \in L$ $\Rightarrow$ $\exists\ \Pi$ (= satisfying assign. to ψ)

  $Pr[V(\Pi) = \text{Accept}] = 1$.

  $x \notin L$ $\Rightarrow$ $OPT(\psi) \le \alpha_0$

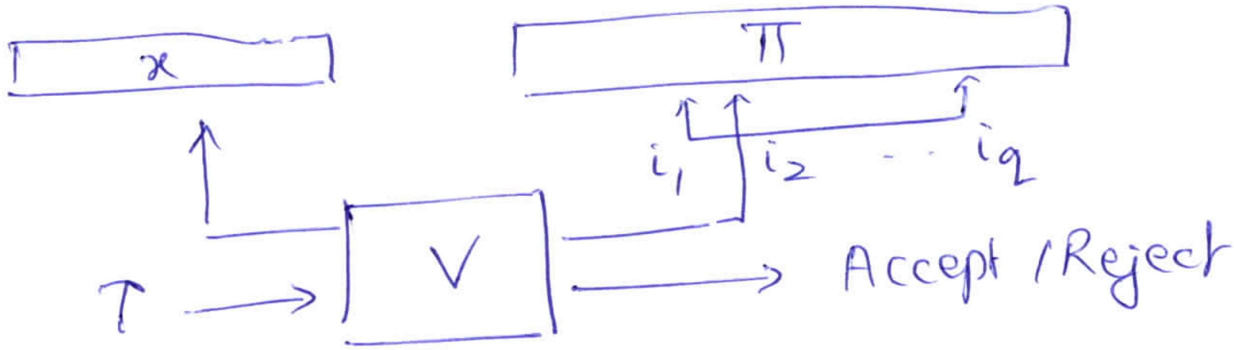
  $\Rightarrow$ $\forall\ \Pi$ (= assignment)

  $Pr[V(\Pi) = \text{Accept}] \le \alpha_0 < 1$.

"soundness" can be boosted to $\frac{1}{2}$ by $O(1)$-repetition ▨

Version #2 $\Rightarrow$ Version #1

- $L \in NP$. We design the reduction.
- We have the verifier



- Think of $\underbrace{\Pi = y_1 y_2 y_3 \cdots y_{|\Pi|}}_{\text{unknown bits / variables}}$
- $\forall \tau$ write the constraint

$$C_\tau \left(y_{i_1(\tau)}, y_{i_2(\tau)}, \ldots, y_{i_q(\tau)} \right).$$

- \# constraints $=$ poly(n). $\boxed{\text{CSP}}$

(Comp). $x \in L \Rightarrow \exists \Pi \quad \Pr[V(\Pi) = \text{Acc}] = 1$.

$\Rightarrow \text{OPT}(\text{C.S.P.}) = 1$.

(Sound.) $x \notin L \Rightarrow \forall \Pi \quad \Pr[V(\Pi) = \text{Acc}] \leq \frac{1}{2}$

$\Rightarrow \text{OPT}(\text{C.S.P.}) \leq \frac{1}{2}$.

This gives a reduction

$$L \longrightarrow \text{C.S.P.}$$

$$x \rightsquigarrow \Gamma$$

$$x \in L \Rightarrow \mathrm{OPT}(\Gamma) = 1$$

$$x \notin L \Rightarrow \mathrm{OPT}(\Gamma) \leq \frac{1}{2}.$$

Every CSP reduces to 3SAT !

$$\text{C.S.P.} \longrightarrow \text{3SAT.}$$

$$\Gamma \rightsquigarrow \Psi.$$

$$\mathrm{OPT}(\Gamma) = 1 \Rightarrow \mathrm{OPT}(\Psi) = 1$$

$$\mathrm{OPT}(\Gamma) \leq \frac{1}{2} \Rightarrow \mathrm{OPT}(\Psi) \leq 1 - \frac{1}{2^q}.$$

q = <u>arity</u> of C.S.P.

This gives the $L \to$ Gap3SAT reduction. $\blacksquare$