Dictatorship Test towards 3SAT

- Since $c = \pm b$ and b is uniform and independent of a, c is also uniform and independent of a.
- We have $\mathbf{Pr}[bc = 1] = \mathbf{Pr}[c = b] = \mathbf{Pr}[a = -1] \cdot (1 - \delta) = (1 - \delta)/2$, and $\mathbf{Pr}[bc = -1] = (1 + \delta)/2$, and thus $\mathbf{E}[bc] = -\delta$.
- If $a = 1$, then $b = -c$, and thus $abc = -1$ (and also $(a, b, c) \neq (1, 1, 1)$). Hence $\mathbf{Pr}[abc = 1] = \mathbf{Pr}[a = -1] \cdot \mathbf{Pr}[b \neq c | a = -1] = \delta/2$, and $\mathbf{Pr}[abc = -1] = 1 - \delta/2$, implying $\mathbf{E}[abc] = \delta - 1$.

The completeness: If $f = \mathrm{Dict}_i$, then $f(x) \vee f(y) \vee f(z)$ iff $(x_i, y_i, z_i) \neq (1, 1, 1)$, which always holds for (x_i, y_i, z_i) chosen as above.

Dictatorship Test towards 3SAT, expectations

$$\mathbf{E}_x[f(x)] = \sum_{\alpha} \hat{f}_\alpha \mathbf{E}_x[\chi_\alpha(x)] = \hat{f}_\emptyset = 0,$$

and similarly
$\mathbf{E}_y[f(y)] = \mathbf{E}_z[f(z)] = \mathbf{E}_{x,y}[f(x)f(y)] = \mathbf{E}_{x,z}[f(x)f(z)] = 0.$

$$\begin{aligned}
\mathbf{E}_{y,z}[f(y)f(z)] &= \sum_{\alpha,\beta} \hat{f}_\alpha \hat{f}_\beta \mathbf{E}_{y,z}[\chi_\alpha(y)\chi_\beta(z)] \\
&= \sum_{\alpha,\beta} \hat{f}_\alpha \hat{f}_\beta \prod_{i\in\alpha\setminus\beta} \mathbf{E}[y_i] \prod_{i\in\beta\setminus\alpha} \mathbf{E}[z_i] \prod_{i\in\alpha\cap\beta} \mathbf{E}[y_i z_i] \\
&= \sum_{\alpha} \hat{f}_\alpha^2 (-\delta)^{|\alpha|}.
\end{aligned}$$

Dictatorship Test towards 3SAT, more expectations

$$\begin{aligned}
\mathbf{E}_{x,y,z}[f(x)f(y)f(z)] &= \sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \mathbf{E}_{x,y,z}[\chi_\alpha(x)\chi_\beta(y)\chi_\gamma(z)] \\
&= \sum_{\alpha,\beta,\gamma} \hat{f}_\alpha \hat{f}_\beta \hat{f}_\gamma \prod_i \mathbf{E}_{x_i,y_i,z_i}[(x \restriction \alpha)_i (y \restriction \beta)_i (z \restriction \gamma)_i] \\
&= \sum_{\alpha \subseteq \beta} \hat{f}_\alpha \hat{f}_\beta^2 \prod_{i \in \beta \setminus \alpha} \mathbf{E}[y_i z_i] \prod_{i \in \alpha} \mathbf{E}[x_i y_i z_i] \\
&= \sum_{\alpha \subseteq \beta} \hat{f}_\alpha \hat{f}_\beta^2 (-\delta)^{|\beta|-|\alpha|} (\delta - 1)^{|\alpha|}.
\end{aligned}$$

Dictatorship Test towards 3SAT, the soundness

$$\begin{aligned} T &= (8 - (1 + f(x))(1 + f(y))(1 + f(z)))/8 \\ &= (7 - f(x) - f(y) - f(z) - f(x)f(y) - f(x)f(z) - f(y)f(z) \\ &\quad - f(x)f(y)f(z))/8 \end{aligned}$$

$$\begin{aligned} 8\varepsilon &\le 8\mathbf{Pr}_{x,y,z}[\text{accepts}] - 7 = 8\mathbf{E}_{x,y,z}\left[T\right] - 7 \\ &= -\mathbf{E}_{x,y,z}[f(x) + f(y) + f(z) + f(x)f(y) + f(x)f(z)] \\ &\quad - \mathbf{E}_{y,z}[f(y)f(z)] - \mathbf{E}_{x,y,z}[f(x)f(y)f(z)] \\ &= -\sum_{\alpha} \hat{f}_\alpha^2 (-\delta)^{|\alpha|} - \sum_{\alpha \subseteq \beta} \hat{f}_\alpha \hat{f}_\beta^2 (-\delta)^{|\beta| - |\alpha|} (\delta - 1)^{|\alpha|}. \end{aligned}$$

- C such that $(1-2\delta+2\delta^2)^C \le \varepsilon^2$.
- γ such that $2^C\gamma \le \varepsilon$.

If $|\hat{f}_\alpha| \le \gamma$ for every α such that $|\alpha| < C$:

$$\Big|\sum_{\alpha\subseteq\beta} \hat{f}_\alpha \hat{f}_\beta^2 (-\delta)^{|\beta|-|\alpha|}(\delta-1)^{|\alpha|}\Big|$$

$$\le \sum_{\alpha\subseteq\beta,|\beta|<C} |\hat{f}_\alpha|\hat{f}_\beta^2 \delta^{|\beta|-|\alpha|}(1-\delta)^{|\alpha|} + \sum_{\alpha\subseteq\beta,|\beta|\ge C} |\hat{f}_\alpha|\hat{f}_\beta^2 \delta^{|\beta|-|\alpha|}(1-\delta)^{|\alpha|}$$

$$\le \max_{|\beta|<C} \sum_{\alpha\subseteq\beta} |\hat{f}_\alpha|\delta^{|\beta|-|\alpha|}(1-\delta)^{|\alpha|} + \max_{|\beta|\ge C} \sum_{\alpha\subseteq\beta} |\hat{f}_\alpha|\delta^{|\beta|-|\alpha|}(1-\delta)^{|\alpha|}$$

$$\le 2^C\gamma + \max_{|\beta|\ge C} \sqrt{\sum_{\alpha\subseteq\beta} \hat{f}_\alpha^2}\sqrt{\sum_{\alpha\subseteq\beta} (\delta^2)^{|\beta|-|\alpha|}((1-\delta)^2)^{|\alpha|}}$$

$$\le \varepsilon + \max_{|\beta|\ge C} \sqrt{(\delta^2+(1-\delta)^2)^{|\beta|}} \le 2\varepsilon.$$

Similarly (easier) $\left|\sum_\alpha \hat{f}_\alpha^2(-\delta)^{|\alpha|}\right| \le 2\varepsilon$, a contradiction.

Max-Cut on Almost Bipartite Graphs (Hardness)

From lecture notes 8: For every $\rho \in (-1, 0)$,

- GapMax-Cut$_{(1-\rho)/2-\varepsilon/2,\pi^{-1}\arccos\rho+\varepsilon/2}$ is NP-hard.

For $\rho = \varepsilon - 1$:

- GapMax-Cut$_{1-\varepsilon,\pi^{-1}\arccos(\varepsilon-1)+\varepsilon/2}$ is NP-hard.

We have

$$\lim_{\varepsilon\to 0} \frac{\pi - \arccos(\varepsilon - 1)}{\sqrt{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{1/\sqrt{1-(1-\varepsilon)^2}}{\varepsilon^{-1/2}/2}$$
$$= 2\sqrt{\lim_{\varepsilon\to 0} \frac{\varepsilon}{2\varepsilon - \varepsilon^2}} = \sqrt{2}.$$

Hence, $\pi^{-1}\arccos(\varepsilon - 1) \leq 1 - \sqrt{\varepsilon}/3 - \varepsilon/2$ for sufficiently small $\varepsilon > 0$, and

- GapMax-Cut$_{1-\varepsilon,1-\sqrt{\varepsilon}/3}$ is NP-hard.

Max-Cut on Almost Bipartite Graphs (Algorithm)

We have $1 - \varepsilon \leq \mathrm{OPT}(G) \leq \mathrm{SDP}(G)$, where

$$\mathrm{SDP}(G) = \frac{1}{|E(G)|} \max \sum_{uv \in E(G)} \frac{1 - \langle x_u | x_v \rangle}{2}$$

over all choices of vectors x_v such that $\|x_v\|_2 = 1$ for $v \in V(G)$.
For this optimal solution,

$$\frac{1}{|E(G)|} \sum_{uv \in E(G)} \langle x_u | x_v \rangle = 1 - 2\mathrm{SDP}(G) \leq 2\varepsilon - 1.$$

Let $E' = \{uv \in E(G) : \langle x_u | x_v \rangle \leq 0\}$; note $|E'| \geq (1 - 2\varepsilon)|E(G)|$.
By lecture notes 5, SDP solution gives a cut of relative size

$$\mathrm{ALG}(G) = \frac{1}{\pi |E(G)|} \sum_{uv \in E(G)} \arccos\langle x_u | x_v \rangle \geq \frac{1}{\pi |E(G)|} \sum_{uv \in E'} \arccos\langle x_u | x_v \rangle$$

$$\geq \frac{1 - 2\varepsilon}{\pi |E'|} \sum_{uv \in E'} \arccos\langle x_u | x_v \rangle$$

$$E' = \{uv \in E(G) : \langle x_u | x_v \rangle \leq 0\}$$

$$2\varepsilon - 1 \geq \frac{1}{|E(G)|} \sum_{uv \in E(G)} \langle x_u | x_v \rangle \geq \frac{1}{|E'|} \sum_{uv \in E'} \langle x_u | x_v \rangle$$

Since $\arccos x$ is convex and decreasing on $[-1, 0]$, we have

$$\begin{aligned}
\mathrm{ALG}(G) &\geq \frac{1 - 2\varepsilon}{\pi |E'|} \sum_{uv \in E'} \arccos \langle x_u | x_v \rangle \\
&\geq \frac{1 - 2\varepsilon}{\pi} \arccos \frac{1}{|E'|} \sum_{uv \in E'} \langle x_u | x_v \rangle \\
&\geq \frac{1 - 2\varepsilon}{\pi} \arccos(2\varepsilon - 1) = 1 - \Theta(\sqrt{\varepsilon})
\end{aligned}$$

Max-Cut to 2SAT

For each edge uv, we add clauses $(u \vee v)$ and $(\neg u \vee \neg v)$.
m edges $\rightarrow n = 2m$ clauses.

- Cut of size αm:
  - Set variables on one side to true, on the other to false.
  - $2\alpha m = \alpha n$ satisfied clauses.
- At least βm un-cut edges gives at least $\beta m = \frac{\beta}{2} n$ unsatisfied clauses.

GapMax-Cut$_{1-\varepsilon,1-\sqrt{\varepsilon}/3}$ is NP-hard $\Rightarrow$ Gap2SAT$_{1-\varepsilon,1-\sqrt{\varepsilon}/6}$ is NP-hard.